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# Three-dimensional $C$ -, $S$ - and $E$ -transforms

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## Abstract

Three-dimensional continuous and discrete Fourier-like transforms, based on the three simple and four semisimple compact Lie groups of rank 3, are presented. For each simple Lie group, there are three families of special functions ( $C$ -,  $S$ - and  $E$ -functions) on which the transforms are built. Pertinent properties of the functions are described in detail, such as their orthogonality within each family, when integrated over a finite region  $F$  of the three-dimensional Euclidean space (continuous orthogonality), as well as when summed up over a lattice grid  $F_M \subset F$  (discrete orthogonality). The positive integer  $M$  sets up the density of the lattice containing  $F_M$ . The expansion of functions given either on  $F$  or on  $F_M$  is the paper's main focus.

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## 1. Introduction

New  $n$ -dimensional  $C$ -,  $S$ - and  $E$ -transforms were recently described in [11–13]. Each transform is based on a compact semisimple Lie group of rank  $n$  and comes in three versions: analogs of Fourier series, Fourier integrals and Fourier transforms on an  $n$ -dimensional lattice. They are named  $C$ -,  $S$ - and  $E$ -transforms [22] in recognition of the fact that they can be understood as generalizations of the one-dimensional cosine, sine and exponential Fourier transform.

The aim of this paper is to set the grounds for the three-dimensional exploitation of transforms—described here as continuous transforms in a finite region  $F$  of the three-dimensional Euclidean space  $\mathbb{R}^3$ , and also as discrete transforms of functions given on a lattice grid of points  $F_M \subset F$  of any density—in ready-to-use form. A positive integer  $M$  specifies the density. In some cases, the density of the grid is more flexible as is dictated by not one, but two or three positive integers. The grid could thus be made denser along certain

axes. The symmetry of the lattice is dictated by the shape of  $F$ , or equivalently, by the choice of the Lie group.

There are seven compact semisimple Lie groups of rank 3:

$$\begin{aligned} &SU(2) \times SU(2) \times SU(2), & SU(3) \times SU(2), & O(5) \times SU(2), \\ &G(2) \times SU(2), & SU(4), & O(7), & Sp(6). \end{aligned}$$

Throughout the paper we identify these cases by symbols that are often used for their respective Lie algebras:

$$A_1 \times A_1 \times A_1, \quad A_2 \times A_1, \quad C_2 \times A_1, \quad G_2 \times A_1, \quad A_3, \quad B_3, \quad C_3.$$

The immediate motivation for this paper is our anticipation of the extensive use of the transforms given the need for processing the rapidly increasing amount of 3D digital data gathered today. In 2D, our group transforms offered only in some cases more than a marginal advantage, having emerged when satisfactory practical methods had already been developed and adequately implemented. So far, practical use of the functions in 2D rested on the fact that the continuous extension of the transformed lattice data displayed remarkably smooth interpolation between lattice points [4] (see also references therein).

Special functions, which serve as the kernel of our transform (we call them  $C$ -,  $S$ - and  $E$ -functions or orbit functions), have simple symmetry property under the action of the corresponding affine Weyl group. The affine group contains as a subgroup the group of translations in  $\mathbb{R}^n$ , which underlies the common Fourier transform. This is the primary reason for the superior performance of our transforms, although detailed comparisons, rather than examples, will have to provide quantitative content to substantiate such a claim.

Other properties of the  $C$ -,  $S$ - and  $E$ -functions are not less important.

Within each family, functions are described in a uniform way for semisimple Lie groups of any type and rank. In this work, we illustrate this uniformity by considering all seven rank 3 group cases in parallel. The price to pay for the uniformity of methods is having to work with non-orthogonal bases which are not normalized.

The functions are defined in  $\mathbb{R}^n$  and have continuous derivatives of all degrees. Their orthogonality, when integrated over the finite region  $F$  appropriate for each Lie group, was shown in [21]. The discrete orthogonality of  $C$ -functions in  $F_M$  has already been described in [19] and extensively used (see for example [5] and references therein). The completeness of these systems of functions directly follows from the completeness of the system of exponential functions.

A Laplace operator for each Lie group is given in a different set of coordinates. The  $C$ - and  $S$ -functions are its eigenfunctions with known eigenvalues. On the boundary of  $F$ , the  $C$ -functions have a vanishing normal derivative, while  $S$ -functions reach zero at the boundary.

The functions have a number of other useful properties, which can be found in [11–13]. For example, the decomposition of their products into sums, the splitting of functions into as many mutually exclusive congruence classes as is the order of the center of the Lie group. These properties are exposed in more details and in a more general setting in a forthcoming paper [6].

A different but valid viewpoint on some of the special functions presented here, namely, functions symmetrized by the summation of constituent functions over a finite group [18], may turn out to be rather useful. The finite group, in the case of  $C$ - and  $S$ -functions, is the Weyl group of the corresponding semisimple Lie group. In the case of  $E$ -functions, it is the even subgroup of the Weyl group. The Weyl group of  $SU(n)$  is isomorphic to the group  $S_n$  of the permutation of  $n$  elements. This led to the recent implementations in [14, 15], where instead of the Weyl group of  $SU(n)$ , the  $S_n$  group is used, and variables are given relative to

an orthonormal system of coordinates. Furthermore, the even subgroup of  $S_n$  is the alternating group. Related transforms were introduced most recently in [16, 17].

The paper is organized as follows.

In section 2, necessary definitions and properties of Lie groups and algebras are given and discussed. Semisimple Lie groups of rank 3 are considered in detail in section 3. For each of these groups, we lay down the information necessary to construct and use their orbit functions for 3D continuous and discrete transforms. Section 4 is devoted to  $C$ -,  $S$ - and  $E$ -orbit functions and to their pertinent properties. Continuous and discrete orbit-function transforms are presented in section 5. Some problems and possible applications arising in connection with orbit functions are formulated in the conclusion. An example of the application of orbit-function transforms in the case of the group  $SU(2) \times SU(2) \times SU(2)$  is given at the end of the paper.

## 2. Pertinent properties of Lie groups and Lie algebras

The notion of orbit function of  $n$  variables depends essentially on the underlying semisimple Lie group of rank  $n$ . This section is intended to recall some of the standard properties of semisimple Lie groups/Lie algebras in general, and particularly those of rank 3, as well as properties of related Weyl groups. We also fix notation and terminology. Additional information about such Lie groups can be found for example in [2, 8, 10, 23].

### 2.1. Definitions and notations

Let  $\mathbb{R}^n$  be the real Euclidean space spanned by the simple roots of a simple Lie group  $G$  (equivalently, Lie algebra). The basis of the simple roots is hereafter referred to as the  $\alpha$ -basis. An  $\alpha$ -basis is not orthogonal and comprises simple roots of at most two different lengths. If a semisimple  $G$  is not simple, the  $\alpha$ -bases of its simple constituents are pairwise orthogonal.

For important practical (i.e. computational) reasons, it is also advantageous to introduce the basis of fundamental weights, hereafter referred to as the  $\omega$ -basis. Moreover, for Lie groups with simple roots of two different lengths, it is useful to introduce bases dual to  $\alpha$ - and  $\omega$ -bases, denoted here as  $\check{\alpha}$ - and  $\check{\omega}$ -bases respectively. Occasionally it is also useful to work with the orthonormal basis  $\{e_1, e_2, \dots, e_n\}$  of  $\mathbb{R}^n$ . Each subsection contains an explicit elaboration of these bases for the groups we consider.

The Cartan matrix  $C$  of  $G$  provides, in principle, all of the information needed about  $G$ . It is an  $n \times n$  matrix. In particular, it provides the relation between  $\alpha$ - and  $\omega$ -bases:

$$\alpha = C\omega, \quad \omega = C^{-1}\alpha,$$

where

$$C_{ij} = \left( \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \right), \quad i, j \in \{1, 2, \dots, n\}.$$

Here  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathbb{R}^n$ . The length of the long simple roots is determined by an additional convention

$$\langle \alpha_{\text{long}}, \alpha_{\text{long}} \rangle = 2.$$

The dual bases are fixed by the relations

$$\check{\alpha}_i = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}, \quad \check{\omega}_i = \frac{2\omega_i}{\langle \alpha_i, \alpha_i \rangle}, \quad \langle \alpha_i, \check{\omega}_j \rangle = \langle \check{\alpha}_i, \omega_j \rangle = \delta_{ij},$$

where  $\delta_{ij}$  is the Kronecker delta.

The root lattice  $Q$  and the weight lattice  $P$  of  $G$  are formed by all integer linear combinations of the  $\alpha$ -basis and  $\omega$ -basis,

$$Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \dots + \mathbb{Z}\alpha_n, \quad P = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \dots + \mathbb{Z}\omega_n.$$

In general,  $Q \subseteq P$ , but in rank 3 Lie groups the equalities do not occur. Similarly, we can introduce the dual lattices  $\check{Q}$  and  $\check{P}$ .

In the weight lattice  $P$ , we define the cone of dominant weights  $P^+$  and its subset of strictly dominant weights  $P^{++}$

$$P \supset P^+ = \mathbb{Z}^{\geq 0}\omega_1 + \dots + \mathbb{Z}^{\geq 0}\omega_n \supset P^{++} = \mathbb{Z}^{> 0}\omega_1 + \dots + \mathbb{Z}^{> 0}\omega_n.$$

For any simple Lie group  $G$  there is a unique highest root  $\xi$

$$\xi = m_1\alpha_1 + m_2\alpha_2 + \dots + m_n\alpha_n = q_1\check{\alpha}_1 + q_2\check{\alpha}_2 + \dots + q_n\check{\alpha}_n. \tag{1}$$

Coefficients  $m_i$  and  $q_i, i = \overline{1, n}$ , are natural numbers, referred to as marks and comarks respectively. They are well known for all simple Lie groups (see for example [8, 10]).

### 2.2. Weyl groups and their orbits

The Weyl group  $W(G)$  of a semisimple Lie group  $G$  is the finite group generated in  $\mathbb{R}^n$  by reflections in  $n - 1$ -dimensional mirrors (hyperplanes) orthogonal to the simple roots of  $G$  and containing the origin of  $\mathbb{R}^n$ . For a simple root  $\alpha_i, i = \overline{1, n}$  the corresponding reflection  $r_{\alpha_i}$  is given by

$$r_{\alpha_i}x = r_{\alpha_i}x = x - \frac{2\langle x, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i = x - \langle x, \check{\alpha}_i \rangle\alpha_i, \quad x \in \mathbb{R}^n. \tag{2}$$

There is a general method for building Weyl group orbits. In section 3, we limit ourselves to recording the result of its application for all Lie groups of rank  $n = 3$ .

It is assumed that we have fixed a semisimple Lie group of rank 3 and that we consider its weight lattice  $P$ . A  $W$ -orbit can be generated from any point  $(a, b, c) \in \mathbb{R}^3$ , but in this paper we are almost always interested in  $W$ -orbits of points in  $P$ . It is convenient to specify the orbit by its unique point  $(a, b, c) = a\omega_1 + b\omega_2 + c\omega_3$  with nonnegative integer coordinates  $a > 0, b > 0, c > 0$ . We denote such a generic orbit by  $W(a, b, c)$ .

Every group contains the trivial one point orbit  $(0, 0, 0)$ . The number of points of a generic orbit  $|W(a, b, c)|$  is equal to the order  $|W|$  of the Weyl group.

### 2.3. Affine Weyl groups and their fundamental domains

Consider the reflection  $r_\xi$  with respect to the hyperplane containing the origin and orthogonal to the highest root  $\xi$ , see (1)

$$r_\xi x = x - \frac{2\langle x, \xi \rangle}{\langle \xi, \xi \rangle}\xi, \quad x \in \mathbb{R}^n.$$

We extend the set of  $n$  reflections  $r_{\alpha_i}$ , given in (2), generating the Weyl group  $W$ , by one reflection  $r_0$

$$r_0x = r_\xi x + \xi, \quad x \in \mathbb{R}^n.$$

The resulting group transformations of  $\mathbb{R}^n$ , generated by  $n + 1$  reflections  $r_0, r_1, \dots, r_n$ , is referred to as the affine Weyl group  $W^{\text{aff}}$ . The order of  $W^{\text{aff}}$  is infinite.

The fundamental region  $F(G) \subset \mathbb{R}^n$  for any  $W^{\text{aff}}(G)$  is the convex hull of the vertices  $\{0, \frac{\omega_1}{q_1}, \dots, \frac{\omega_n}{q_n}\}$ , where  $q_i, i = \overline{1, n}$  are comarks from (1), or equivalently,

$$F(G) = \left\{ 0, \frac{\check{\omega}_1}{m_1}, \frac{\check{\omega}_2}{m_2}, \dots, \frac{\check{\omega}_n}{m_n} \right\}, \quad \text{where } m_i, i = \overline{1, n} \text{ are marks, see (1).} \tag{3}$$

Repeated reflections of  $F(G)$  in its  $(n - 1)$ -dimensional sides results in tiling the entire space  $\mathbb{R}^n$  by copies of  $F$ . We define grid  $F_M \subset F$ , depending on an arbitrary natural number  $M$  as given in (1),

$$F_M = \left\{ \frac{s_1}{M} \check{\omega}_1 + \frac{s_2}{M} \check{\omega}_2 + \dots + \frac{s_n}{M} \check{\omega}_n \mid s_1, \dots, s_n \in \mathbb{Z}^{\geq 0}, \sum_{i=1}^n s_i m_i \leq M > 0 \right\}.$$

The number of points of the grid  $F_M$  is denoted by  $|F_M|$ , and the volume of the fundamental region  $F$  with respect to the euclidian measure is denoted by  $|F|$ .

**Remark 1.** In case  $G = G' \times G''$ , where  $G'$  and  $G''$  are simple, the fundamental region of  $G$  is the Cartesian product of the fundamental regions of  $G'$  and  $G''$ . The same holds for grids  $F_{M',M''}$ , but for each simple constituent of  $G$ , the numbers  $M'$  and  $M''$  could be chosen independently.

**Remark 2.** In the case of two different numbers  $M_1$  and  $M_2$ , such that  $M_1 \bmod M_2 = 0$ , the corresponding grids for the same semisimple group  $G$  are related as follows:  $F_{M_2} \subset F_{M_1}$ .

When  $G = G' \times G''$  (or  $G = G' \times G'' \times G'''$ , etc.) the same holds for the sets of grids, i.e., if  $M'_1 \bmod M'_2 = 0$  and  $M''_1 \bmod M''_2 = 0$  then  $F_{M'_2, M''_2} \subset F_{M'_1, M''_1}$ .

2.4. Even subgroup  $W^e$  of the Weyl group

Elements of the subgroup  $W^e \subset W$  are formed by an even number of reflections which generate  $W$ . Each  $C$ - and  $S$ -function is built on a single Weyl group orbit of point  $\lambda \in P$ , and each  $E$ -function is built on an orbit of the even subgroup of the Weyl group, defined as follows.

Consider a subgroup of  $W$  generated by an even number of reflections

$$W_e = \{r_{i_1} r_{i_2} \dots r_{i_{2k}} \mid k \in \mathbb{N}, r_{i_l} \in W, l = \overline{1, 2k}\}. \tag{4}$$

$W_e$  is a normal subgroup of the index 2 of the Weyl group, i.e.  $2|W_e| = |W|$ . Let us denote the orbit of point  $\lambda \in P$  with respect to the action of  $W_e$  by  $W_e(\lambda)$ , and the size of this orbit by  $|W_e(\lambda)|$ , then

$$W(\lambda) = \begin{cases} W_e(\lambda) \cup W_e(r_i \lambda), \text{ for some } r_i \in W, & \text{when } \lambda \in P^{++} \\ W_e(\lambda), & \text{when } \lambda \in P^+ \setminus P^{++} \end{cases} \tag{5}$$

and one of the following relations holds true  $|W_e(\lambda)| = \frac{1}{2}|W(\lambda)|$ , when  $\lambda \in P^{++}$  or  $|W_e(\lambda)| = |W(\lambda)|$ , when  $\lambda \in P^+ \setminus P^{++}$ . Note that each orbit of the even subgroup  $W_e$  contains exactly one point from  $P_e := P^+ \cup r_i P^{++}, r_i \in W$ .

Similarly, we define the even affine Weyl subgroup

$$W_e^{\text{aff}} = \{r_{i_1} r_{i_2} \dots r_{i_{2k}} \mid k \in \mathbb{N}, r_{i_l} \in W^{\text{aff}}, l = \overline{1, 2k}\},$$

and its fundamental region

$$F_e = F \cup r_i F, \tag{6}$$

where  $F$  is a fundamental region of  $W^{\text{aff}}$  (3) and  $r_i \in r_0, r_1, \dots, r_n$ .

The same formula holds true for the grid on the fundamental region  $F_e$

$$F_{eM} = F_M \cup r_i F_M, \quad r_i \in W^{\text{aff}}.$$

**Remark 3.** As it follows from (6), the fundamental region of  $W_e$  is not unique, hence it can be chosen in such a way as to be convenient for a given application.

**Example 1.** Consider the rank one compact simple Lie group  $SU(2)$  (the corresponding Lie algebra is  $A_1$ ). The Cartan matrix in this case is the  $1 \times 1$  matrix  $C = (2)$ . The root system consists of two roots  $\pm\alpha$ . The root and weight lattices are formed by integer multiples of the simple root and integer multiples of the fundamental weight  $\omega$ .

$$Q = \{\mathbb{Z}\alpha\}, \quad P = \{\mathbb{Z}\omega\}, \quad \text{where } \alpha = C\omega \text{ so that } \omega = \frac{1}{2}\alpha.$$

Therefore  $P = Q \cup (Q + \omega)$ .

The Weyl group  $W$  of  $A_1$  is of order 2. It is a reflection group acting in  $\mathbb{R}$ . We have  $W = \{1, -1\}$  and  $W_e = \{1\}$ . Consequently, a Weyl group orbit containing the point  $x \neq 0$ , also contains the point  $-x$ , but the orbit of the even subgroup of  $W$  consists of a single point, either  $x$  or  $-x$ .

The fundamental region  $F$  is the segment with endpoints  $F = \{0, \omega\}$ .

The grid  $F_M \subset F$  is fixed by the positive integer  $M$  and it consists of  $M + 1$  points

$$F_M = \left\{0, \frac{1}{M}, \frac{2}{M}, \dots, \frac{M-1}{M}, 1\right\}.$$

### 3. Semisimple Lie algebras of rank 3

In this section, we provide specific information about compact semisimple Lie groups of rank 3, namely  $SU(2) \times SU(2) \times SU(2)$ ,  $O(5) \times SU(2)$ ,  $SU(3) \times SU(2)$ ,  $G_2 \times SU(2)$ , and the simple groups  $Sp(6)$ ,  $O(7)$  and  $SU(4)$ . For each of the groups, we lay down all of the information necessary to construct and use their orbit functions for 3D continuous and discrete transforms.

Note, that only the weight lattices of the Lie groups  $SU(2) \times SU(2) \times SU(2)$ ,  $O(5) \times SU(2)$ ,  $O(7)$  and  $Sp(6)$  display cubic symmetries.

Below we present the Dynkin diagram for each semisimple Lie group of rank 3. On these diagrams, long and short simple roots  $\alpha$  are respectively denoted by unfilled and filled circles and comarks are presented over the circles.

#### 3.1. The Lie algebra $A_1 \times A_1 \times A_1$

This case is a straightforward concatenation of three copies of  $A_1$  (see example 1). The Dynkin diagram and Cartan matrix with its inverse are the following,

$$\begin{array}{ccc} \textcircled{1} & \textcircled{1} & \textcircled{1} \\ \alpha_1 & \alpha_2 & \alpha_3 \end{array} \quad C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad C^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

All simple roots of the same length equal to  $\sqrt{2}$ . The bases of simple roots and fundamental weights are thus related by

$$\alpha_i = 2\omega_i, \quad \omega_i = \frac{1}{2}\alpha_i, \quad \check{\alpha}_1 = \alpha_1, \quad \check{\omega}_1 = \omega_1, \quad i = 1, 2, 3.$$

These bases are easily written in the orthonormal basis  $\{e_1, e_2, e_3\}$

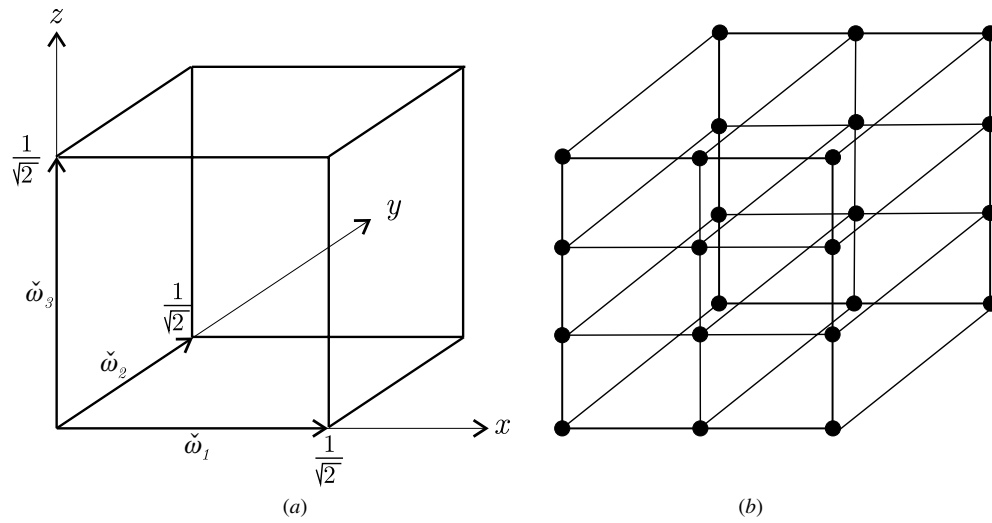
$$\alpha_i = \sqrt{2}e_i, \quad \omega_i = \frac{1}{\sqrt{2}}e_i, \quad \check{\omega}_i = \frac{1}{\sqrt{2}}e_i, \quad i = 1, 2, 3.$$

The highest roots for the simple subgroups  $A_1$  are given by the formulae

$$\xi = \alpha_1, \quad \xi = \alpha_2, \quad \xi = \alpha_3.$$

The fundamental region is a cube with vertices, see figure 1(a)

$$F(A_1 \times A_1 \times A_1) = \{0, \omega_i, \omega_i + \omega_j, \omega_1 + \omega_2 + \omega_3\}, \quad \text{where } j < i \in 1, 2, 3.$$



**Figure 1.** (a) the Fundamental region  $F$  of the Lie algebra  $A_1 \times A_1 \times A_1$ ;  $x, y$  and  $z$  indicate respectively the orthogonal directions of the orthonormal basis  $\{e_1, e_2, e_3\}$ ; (b) the grid  $F_{2,1,3}(A_1 \times A_1 \times A_1)$ .

The volume of the fundamental region is given by the formula

$$|F(A_1 \times A_1 \times A_1)| = |\check{\omega}_1| \cdot |\check{\omega}_2| \cdot |\check{\omega}_3| = \frac{1}{2\sqrt{2}}.$$

The grid  $F_{M,M',M''} \subset F$  is fixed by the independent choice of three positive integers  $M, M'$  and  $M''$ , and consists of all the points

$$F_{M,M',M''}(A_1 \times A_1 \times A_1) = \left\{ \frac{s_1}{M} \check{\omega}_1 + \frac{s'_1}{M'} \check{\omega}_2 + \frac{s''_1}{M''} \check{\omega}_3 \mid s_1 \leq M, s'_1 \leq M', s''_1 \leq M''; s_1, s'_1, s''_1 \in \mathbb{Z}^{\geq 0} \right\}.$$

The grid is cubic if  $M = M' = M''$ , otherwise it is rectangular. The freedom to use unequal values of  $M, M'$  and  $M''$  may prove rather useful in the analysis of data given on rectangular, but not cubic grids.

The number of points in the grid  $F_{M,M',M''}$  equals

$$|F_{M,M',M''}(A_1 \times A_1 \times A_1)| = |F_M(A_1)| \cdot |F_{M'}(A_1)| \cdot |F_{M''}(A_1)| = (M + 1)(M' + 1)(M'' + 1).$$

**Example 2.** Consider the case  $M = M' = 1, M'' = 3$ . There are 16 points of  $F_{1,1,3}(A_1 \times A_1 \times A_1)$ . Explicitly, we have the following sets of integers  $[s_1, s'_1, s''_1]$  and the corresponding grid points in the  $\check{\omega}$ -basis (in this case the basis is orthogonal),  $(\frac{s_1}{M}, \frac{s'_1}{M'}, \frac{s''_1}{M''})$ :

$$\begin{aligned} [0, 0, 0] &= (0, 0, 0), & [0, 0, 1] &= (0, 0, \frac{1}{3}), & [0, 0, 2] &= (0, 0, \frac{2}{3}), & [0, 0, 3] &= (0, 0, 1), \\ [0, 1, 0] &= (0, 1, 0), & [0, 1, 1] &= (0, 1, \frac{1}{3}), & [0, 1, 2] &= (0, 1, \frac{2}{3}), & [0, 1, 3] &= (0, 1, 1), \\ [1, 0, 0] &= (1, 0, 0), & [1, 0, 1] &= (1, 0, \frac{1}{3}), & [1, 0, 2] &= (1, 0, \frac{2}{3}), & [1, 0, 3] &= (1, 0, 1), \\ [1, 1, 0] &= (1, 1, 0), & [1, 1, 1] &= (1, 1, \frac{1}{3}), & [1, 1, 2] &= (1, 1, \frac{2}{3}), & [1, 1, 3] &= (1, 1, 1). \end{aligned}$$

Next, consider the case  $M = 2, M' = 1, M'' = 3$ , see figure 1(b). The 16 points of  $F_{1,1,3}(A_1 \times A_1 \times A_1)$  form a part of  $F_{2,1,3}(A_1 \times A_1 \times A_1)$ . Note that they correspond to



another set of  $[s_1, s'_1, s''_1]$ . More precisely,  $F_{1,1,3}$  is the subset of  $F_{2,1,3}$  with an even value of  $s_1$ . Thus  $F_{2,1,3}(A_1 \times A_1 \times A_1)$  contains an additional eight points:

$$[1, 0, 0] = \left(\frac{1}{2}, 0, 0\right), \quad [1, 0, 1] = \left(\frac{1}{2}, 0, \frac{1}{3}\right), \quad [1, 0, 2] = \left(\frac{1}{2}, 0, \frac{2}{3}\right), \quad [1, 0, 3] = \left(\frac{1}{2}, 0, 1\right), \\ [1, 1, 0] = \left(\frac{1}{2}, 1, 0\right), \quad [1, 1, 1] = \left(\frac{1}{2}, 1, \frac{1}{3}\right), \quad [1, 1, 2] = \left(\frac{1}{2}, 1, \frac{2}{3}\right), \quad [1, 1, 3] = \left(\frac{1}{2}, 1, 1\right).$$

The Weyl group orbit of the generic point  $a\omega_1 + b\omega_2 + c\omega_3$ ,  $a, b, c > 0$ , always consists of the eight points

$$W_{(a,b,c)}(A_1 \times A_1 \times A_1) = \{(\pm a, \pm b, \pm c)\}.$$

Orbit sizes for arbitrary points are given by the relations

$$|W_{(a,b,c)}| = 8, \quad |W_{(a,b,0)}| = |W_{(a,0,c)}| = |W_{(0,b,c)}| = 4, \\ |W_{(0,0,0)}| = 1, \quad |W_{(a,0,0)}| = |W_{(0,b,0)}| = |W_{(0,0,c)}| = 2.$$

### 3.2. The Lie algebra $A_2 \times A_1$

The Dynkin diagram and Cartan matrix with its inverse are the following,

$$\begin{array}{ccc} 1 & 1 & 1 \\ \circ & \circ & \circ \\ \alpha_1 & \alpha_2 & \alpha_3 \end{array} \quad C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad C^{-1} = \frac{1}{6} \begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Hence all the simple roots of the same length equal to  $\sqrt{2}$ .

The bases of simple roots and fundamental weights are thus related by

$$\alpha_1 = 2\omega_1 - \omega_2, \quad \omega_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, \quad \check{\alpha}_1 = \alpha_1, \quad \check{\omega}_1 = \omega_1, \\ \alpha_2 = -\omega_1 + 2\omega_2, \quad \omega_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2, \quad \check{\alpha}_2 = \alpha_2, \quad \check{\omega}_2 = \omega_2, \\ \alpha_3 = 2\omega_3; \quad \omega_3 = \frac{1}{2}\alpha_3, \quad \check{\alpha}_3 = \alpha_3, \quad \check{\omega}_3 = \omega_3.$$

In order to visualize the implied geometry, it is useful to represent the  $\omega$  and  $\alpha$  bases in the orthonormal basis. We have

$$\alpha_1 = (1, -1, 0) = e_1 - e_2, \quad \omega_1 = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right) = \frac{2}{3}e_1 - \frac{1}{3}e_2 - \frac{1}{3}e_3 = \check{\omega}_1, \\ \alpha_2 = (0, 1, -1) = e_2 - e_3, \quad \omega_2 = \left(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}\right) = \frac{1}{3}e_1 + \frac{1}{3}e_2 - \frac{2}{3}e_3 = \check{\omega}_2, \\ \alpha_3 = \frac{\sqrt{2}}{\sqrt{3}}(1, 1, 1) = \frac{\sqrt{2}}{\sqrt{3}}(e_1 + e_2 + e_3), \quad \omega_3 = \frac{1}{\sqrt{6}}(1, 1, 1) = \frac{1}{\sqrt{6}}(e_1 + e_2 + e_3) = \check{\omega}_3.$$

The highest roots for the simple subgroups  $A_2$  and  $A_1$  are given by the formulae

$$\xi = \alpha_1 + \alpha_2, \quad \xi = \alpha_3.$$

The fundamental region  $F$  is a cylinder (see figure 2(a)) with an equilateral triangle as its base and  $\omega_3$  as its height. Its vertices are

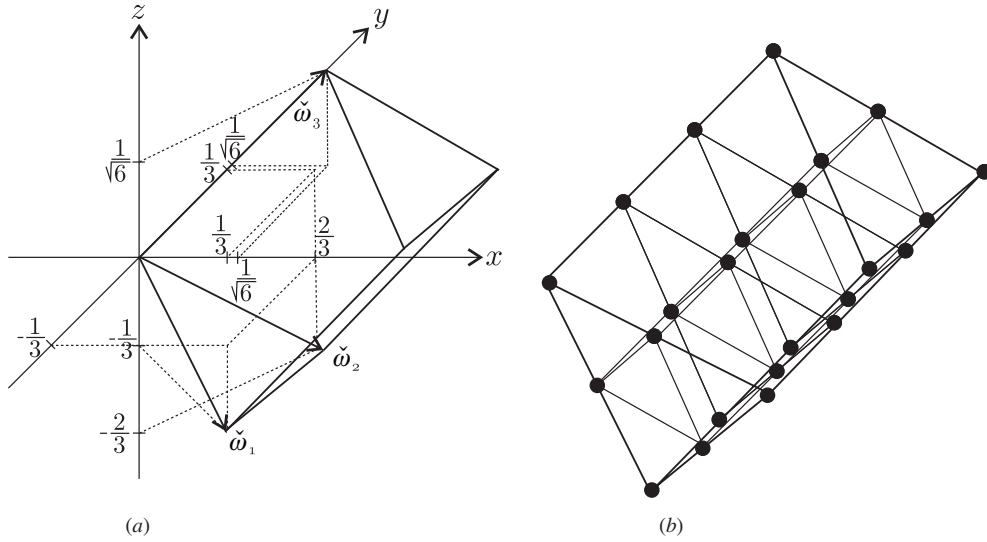
$$F(A_2 \times A_1) = \{0, \omega_1, \omega_2, \omega_3, \omega_1 + \omega_3, \omega_2 + \omega_3\}.$$

The volume of the fundamental region is given by the formula

$$|F(A_2 \times A_1)| = \frac{1}{2}([\check{\omega}_1, \check{\omega}_2], \check{\omega}_3) = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{\sqrt{6}} \begin{vmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2\sqrt{6}}.$$

The grid  $F_{M,M'}(A_2 \times A_1) \subset F(A_2 \times A_1)$  is fixed by the independent choice of the two positive integers  $M$  and  $M'$ . It consists of all the points

$$F_{M,M'}(A_2 \times A_1) = \left\{ \frac{s_1}{M}\check{\omega}_1 + \frac{s_2}{M}\check{\omega}_2 + \frac{s'_1}{M'}\check{\omega}_3 \mid s_1 + s_2 \leq M, s'_1 \leq M'; s_1, s_2, s'_1 \in \mathbb{Z}^{\geq 0} \right\}.$$



**Figure 2.** (a) The fundamental region  $F$  of the Lie algebra  $A_2 \times A_1$ ;  $x$ ,  $y$  and  $z$  indicate respectively the orthogonal directions of the orthonormal basis  $\{e_1, e_2, e_3\}$  and the fundamental region of  $A_2$  lies in the plane  $x + y + z = 0$ ; (b) the grid  $F_{2,3}(A_2 \times A_1)$ .

The number of points in the grid  $F_{M,M'}(A_2 \times A_1)$  equals

$$|F_{M,M'}(A_2 \times A_1)| = |F_M(A_2)| \cdot |F_{M'}(A_1)| = \left( (M+1)^2 - \frac{M(M+1)}{2} \right) (M'+1).$$

**Example 3.** Consider the case  $M = 2, M' = 3$ , see figure 2(b). There exist 24 points of  $F_{2,3}$

$$|F_{2,3}(A_2 \times A_1)| = \left( (2+1)^2 - \frac{2(2+1)}{2} \right) (3+1) = 24.$$

Explicitly, we have the following sets of integers  $[s_1, s_2, s'_1]$  and the corresponding grid points  $(\frac{s_1}{M}, \frac{s_2}{M}, \frac{s'_1}{M'})$  in  $\check{\omega}$ -basis:

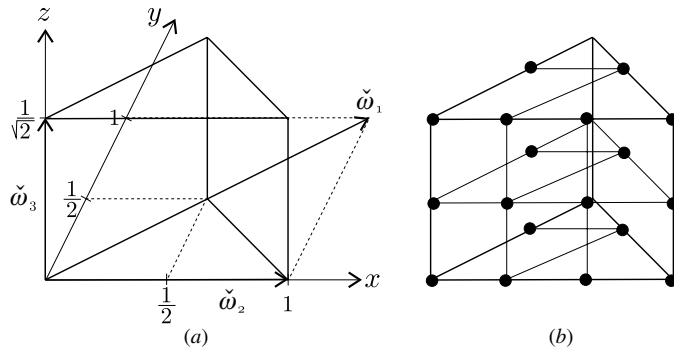
$$\begin{aligned} [0, 2, 0] &= (0, 1, 0), & [0, 2, 3] &= (0, 1, 1), & [0, 2, 2] &= (0, 1, \frac{2}{3}), & [0, 2, 1] &= (0, 1, \frac{1}{3}), \\ [2, 0, 0] &= (1, 0, 0), & [2, 0, 3] &= (1, 0, 1), & [2, 0, 2] &= (1, 0, \frac{2}{3}), & [2, 0, 1] &= (1, 0, \frac{1}{3}), \\ [0, 0, 0] &= (0, 0, 0), & [0, 0, 3] &= (0, 0, 1), & [0, 0, 2] &= (0, 0, \frac{2}{3}), & [0, 0, 1] &= (0, 0, \frac{1}{3}), \\ [0, 1, 0] &= (0, \frac{1}{2}, 0), & [0, 1, 3] &= (0, \frac{1}{2}, 1), & [0, 1, 2] &= (0, \frac{1}{2}, \frac{2}{3}), & [0, 1, 1] &= (0, \frac{1}{2}, \frac{1}{3}), \\ [1, 0, 0] &= (\frac{1}{2}, 0, 0), & [1, 0, 3] &= (\frac{1}{2}, 0, 1), & [1, 0, 2] &= (\frac{1}{2}, 0, \frac{2}{3}), & [1, 0, 1] &= (\frac{1}{2}, 0, \frac{1}{3}), \\ [1, 1, 0] &= (\frac{1}{2}, \frac{1}{2}, 0), & [1, 1, 3] &= (\frac{1}{2}, \frac{1}{2}, 1), & [1, 1, 2] &= (\frac{1}{2}, \frac{1}{2}, \frac{2}{3}), & [1, 1, 1] &= (\frac{1}{2}, \frac{1}{2}, \frac{1}{3}). \end{aligned}$$

The Weyl group orbit of the generic point  $a\omega_1 + b\omega_2 + c\omega_3, a, b, c > 0$ , consists of 12 points

$$W_{(a,b,c)}((A_2 \times A_1)) = \{(a, b, \pm c), (-a, a+b, \pm c), (a+b, -b, \pm c), (b, -(a+b), \pm c), (-(a+b), a, \pm c), (-b, -a, \pm c)\}.$$

Orbit sizes for arbitrary points are given by the relations

$$\begin{aligned} |W_{(a,b,c)}| &= 12, & |W_{(a,b,0)}| &= 6, & |W_{(a,0,c)}| &= 6, & |W_{(0,b,c)}| &= 6, \\ |W_{(a,0,0)}| &= 3, & |W_{(0,b,0)}| &= 3, & |W_{(0,0,c)}| &= 2, & |W_{(0,0,0)}| &= 1. \end{aligned}$$



**Figure 3.** (a) The fundamental region  $F$  of the Lie algebra  $C_2 \times A_1$ ;  $x$ ,  $y$  and  $z$  indicate respectively the orthogonal directions of the orthonormal basis  $\{e_1, e_2, e_3\}$  and the fundamental region of  $C_2$  is the isosceles right-angled triangle in the plane  $z = 0$ , its right angle is at the point  $\frac{1}{2}\check{\omega}_1$ ; (b) the grid  $F_{3,2}(C_2 \times A_1)$ .

### 3.3. The Lie algebra $C_2 \times A_1$

The Dynkin diagram and Cartan matrix with its inverse are the following,

$$\begin{array}{ccc}
 \bullet & \circ & \circ \\
 \alpha_1 & \alpha_2 & \alpha_3
 \end{array}
 \quad
 C = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},
 \quad
 C^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence  $\alpha_1$  is the shorter of the simple roots. The relative lengths of the simple roots are set as  $\langle \alpha_1, \alpha_1 \rangle = 1$  and  $\langle \alpha_2, \alpha_2 \rangle = \langle \alpha_3, \alpha_3 \rangle = 2$ .

The bases of simple roots and fundamental weights are thus related by

$$\begin{array}{llll}
 \alpha_1 = 2\omega_1 - \omega_2, & \omega_1 = \alpha_1 + \frac{1}{2}\alpha_2, & \check{\alpha}_1 = 2\alpha_1, & \check{\omega}_1 = 2\omega_1, \\
 \alpha_2 = -2\omega_1 + 2\omega_2, & \omega_2 = \alpha_1 + \alpha_2, & \check{\alpha}_2 = \alpha_2, & \check{\omega}_2 = \omega_2, \\
 \alpha_3 = 2\omega_3; & \omega_3 = \frac{1}{2}\alpha_3; & \check{\alpha}_3 = \alpha_3; & \check{\omega}_3 = \omega_3.
 \end{array}$$

In the orthonormal basis these bases have the form

$$\begin{array}{lll}
 \alpha_1 = (0, 1, 0) = e_2, & \omega_1 = (\frac{1}{2}, \frac{1}{2}, 0) = \frac{1}{2}e_1 + \frac{1}{2}e_2, & \check{\omega}_1 = (1, 1, 0), \\
 \alpha_2 = (1, -1, 0) = e_1 - e_2, & \omega_2 = (1, 0, 0) = e_1, & \check{\omega}_2 = (1, 0, 0), \\
 \alpha_3 = (0, 0, \sqrt{2}) = \sqrt{2}e_3; & \omega_3 = (0, 0, \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}e_3; & \check{\omega}_3 = (0, 0, \frac{1}{\sqrt{2}}).
 \end{array}$$

The highest roots for the simple subgroups  $C_2$  and  $A_1$  are given by the formulae

$$\xi = 2\alpha_1 + \alpha_2, \quad \xi = \alpha_3.$$

The fundamental region  $F(C_2 \times A_1)$  is a cylinder (see figure 3(a)) with a triangular base. Its vertices are

$$F(C_2 \times A_1) = \{0, \frac{1}{2}\check{\omega}_1, \check{\omega}_2, \check{\omega}_3, \frac{1}{2}\check{\omega}_1 + \check{\omega}_3, \check{\omega}_2 + \check{\omega}_3\}.$$

The volume of the fundamental region is given by the formula

$$|F(C_2 \times A_1)| = \frac{1}{4}([\check{\omega}_2, \check{\omega}_1], \check{\omega}_3) = \frac{1}{4} \cdot \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{4\sqrt{2}}.$$

The grid  $F_{M,M'} \subset F$  is fixed by the independent choice of the two positive integers  $M$  and  $M'$ . It consists of all the points

$$F_{M,M'}(C_2 \times A_1) = \left\{ \frac{s_1}{M}\check{\omega}_1 + \frac{s_2}{M}\check{\omega}_2 + \frac{s'_1}{M'}\check{\omega}_3 \mid 2s_1 + s_2 \leq M, s'_1 \leq M'; s_1, s_2, s'_1 \in \mathbb{Z}^{\geq 0} \right\}.$$

The number of points in the grid  $F_{M,M'}(C_2 \times A_1)$  equals

$$|F_{M,M'}(C_2 \times A_1)| = |F_M(C_2)| \cdot |F_{M'}(A_1)| = \left( \left[ \frac{M}{2} \right] + 1 \right) \left( M + 1 - \left[ \frac{M}{2} \right] \right) (M' + 1),$$

where  $[\cdot]$  denotes the integer part of a number.

**Example 4.** Consider the case  $M = 3, M' = 2$ . There exist 18 points of  $F_{3,2}(C_2 \times A_1)$ , see figure 3(b)

$$|F_{3,2}(C_2 \times A_1)| = \left( \left[ \frac{3}{2} \right] + 1 \right) \left( 3 + 1 - \left[ \frac{3}{2} \right] \right) (2 + 1) = 18.$$

Explicitly, we have the following sets of integers  $[s_1, s_2, s'_1]$  and the corresponding grid points in the  $\check{\omega}$ -basis,  $(\frac{s_1}{M}, \frac{s_2}{M}, \frac{s'_1}{M'})$ :

$$\begin{aligned} [1, 0, 0] &= \left(\frac{1}{3}, 0, 0\right), & [1, 0, 1] &= \left(\frac{1}{3}, 0, \frac{1}{2}\right), & [1, 0, 2] &= \left(\frac{1}{3}, 0, 2\right), \\ [0, 1, 0] &= \left(0, \frac{1}{3}, 0\right), & [0, 1, 1] &= \left(0, \frac{1}{3}, \frac{1}{2}\right), & [0, 1, 2] &= \left(0, \frac{1}{3}, 2\right), \\ [0, 2, 0] &= \left(0, \frac{2}{3}, 0\right), & [0, 2, 1] &= \left(0, \frac{2}{3}, \frac{1}{2}\right), & [0, 2, 2] &= \left(0, \frac{2}{3}, 2\right), \\ [1, 1, 0] &= \left(\frac{1}{3}, \frac{1}{3}, 0\right), & [1, 1, 1] &= \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{2}\right), & [1, 1, 2] &= \left(\frac{1}{3}, \frac{1}{3}, 2\right), \\ [0, 0, 0] &= (0, 0, 0), & [0, 0, 1] &= \left(0, 0, \frac{1}{2}\right), & [0, 0, 2] &= (0, 0, 2), \\ [0, 3, 0] &= (0, 1, 0), & [0, 3, 1] &= \left(0, 1, \frac{1}{2}\right), & [0, 3, 2] &= (0, 1, 2). \end{aligned}$$

The Weyl group orbit of the generic point  $a\omega_1 + b\omega_2 + c\omega_3, a, b, c > 0$ , consists of the following set of points:

$$\begin{aligned} W_{(a,b,c)}(C_2 \times A_1) &= \{ \pm(a, b, c), \pm(a, b, -c), \\ &\pm(-a, a + b, c), \pm(a + 2b, -b, c), \pm(a + 2b, -(a + b), c), \\ &\pm(-a, a + b, -c), \pm(a + 2b, -(a + b), -c), \pm(a + 2b, -b, -c) \}. \end{aligned}$$

Orbit sizes for arbitrary points are given by the relations

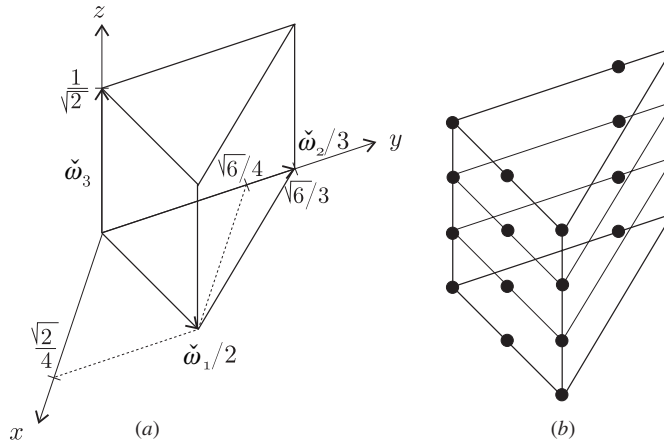
$$\begin{aligned} |W_{(a,b,c)}| &= 16, & |W_{(a,b,0)}| &= 8, & |W_{(a,0,c)}| &= 8, & |W_{(0,b,c)}| &= 8, \\ |W_{(a,0,0)}| &= 4, & |W_{(0,b,0)}| &= 4, & |W_{(0,0,c)}| &= 2, & |W_{(0,0,0)}| &= 1. \end{aligned}$$

### 3.4. The Lie algebra $G_2 \times A_1$

The Dynkin diagram and Cartan matrix with its inverse are the following,

$$\begin{array}{ccc} 2 & 3 & 1 \\ \circ & \bullet & \circ \\ \alpha_1 & \alpha_2 & \alpha_3 \end{array} \quad C = \begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad C^{-1} = \frac{1}{2} \begin{pmatrix} 4 & 6 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence  $\alpha_2$  is the shorter of the simple roots. The relative lengths of the simple roots are set as  $\langle \alpha_2, \alpha_2 \rangle = \frac{2}{3}$  and  $\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_3, \alpha_3 \rangle = 2$ .



**Figure 4.** (a) The fundamental region  $F$  of the Lie algebra  $G_2 \times A_1$ ;  $x$ ,  $y$  and  $z$  indicate respectively the orthogonal directions of the orthonormal basis  $\{e_1, e_2, e_3\}$  and the fundamental region of  $G_2$  is the half of the equilateral triangle in the plane  $z = 0$ ; (b) the grid  $F_{4,3}(G_2 \times A_1)$ .

The bases of simple roots and fundamental weights are thus related by

$$\begin{aligned} \alpha_1 &= 2\omega_1 - 3\omega_2, & \omega_1 &= 2\alpha_1 + 3\alpha_2, & \check{\alpha}_1 &= \alpha_1, & \check{\omega}_1 &= \omega_1, \\ \alpha_2 &= -\omega_1 + 2\omega_2, & \omega_2 &= \alpha_1 + 2\alpha_2, & \check{\alpha}_2 &= 3\alpha_2, & \check{\omega}_2 &= 3\omega_2, \\ \alpha_3 &= 2\omega_3; & \omega_3 &= \frac{1}{2}\alpha_3; & \check{\alpha}_3 &= \alpha_3; & \check{\omega}_3 &= \omega_3. \end{aligned}$$

Relative to the orthonormal basis, we have

$$\begin{aligned} \alpha_1 &= (\sqrt{2}, 0, 0) = \sqrt{2}e_1, & \omega_1 &= \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}, 0\right) = \frac{1}{\sqrt{2}}e_1 + \frac{\sqrt{3}}{\sqrt{2}}e_2, & \check{\omega}_1 &= \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}, 0\right), \\ \alpha_2 &= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, 0\right) = -\frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{6}}e_2, & \omega_2 &= \left(0, \frac{\sqrt{2}}{\sqrt{3}}, 0\right) = \frac{\sqrt{2}}{\sqrt{3}}e_2, & \check{\omega}_2 &= (0, \sqrt{6}, 0), \\ \alpha_3 &= (0, 0, \sqrt{2}) = \sqrt{2}e_3; & \omega_3 &= \left(0, 0, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e_3; & \check{\omega}_3 &= \left(0, 0, \frac{1}{\sqrt{2}}\right). \end{aligned}$$

The highest roots for the simple subgroups are given by the formulae

$$\xi = 2\alpha_1 + 3\alpha_2, \quad \xi = \alpha_3.$$

The fundamental region  $F$  is a cylinder (see figure 4(a)) with a triangular base. Its vertices are

$$F(G_2 \times A_1) = \left\{0, \frac{1}{2}\check{\omega}_1, \frac{1}{3}\check{\omega}_2, \check{\omega}_3, \frac{1}{2}\check{\omega}_1 + \check{\omega}_3, \frac{1}{3}\check{\omega}_2 + \check{\omega}_3\right\}.$$

The volume of the fundamental region is given by the formula

$$|F(G_2 \times A_1)| = \frac{1}{2} \left\langle \left[ \frac{1}{2}\check{\omega}_1, \frac{1}{3}\check{\omega}_2 \right], \check{\omega}_3 \right\rangle = \frac{1}{2} \cdot \frac{1}{2\sqrt{2}} \cdot \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & \sqrt{3} & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{\sqrt{6}}{24},$$

where  $[\cdot, \cdot]$  denotes the vector product.

The grid  $F_{M,M'} \subset F$  is fixed by the independent choice of two positive integers  $M$  and  $M'$ . It consists of all the points

$$F_{M,M'}(G_2 \times A_1) = \left\{ \frac{s_1}{M}\check{\omega}_1 + \frac{s_2}{M}\check{\omega}_2 + \frac{s'_1}{M'}\check{\omega}_3 \mid 2s_1 + 3s_2 \leq M, s'_1 \leq M'; s_1, s_2, s'_1 \in \mathbb{Z}^{\geq 0} \right\}.$$

The number of points in the grid  $F_{M,M'}$  equals

$$|F_{M,M'}(G_2 \times A_1)| = |F_M(G_2)| \cdot |F_{M'}(A_1)| = \left( \left[ \frac{M}{3} \right] + 1 + \sum_{i=0}^{\lfloor \frac{M}{3} \rfloor} \left[ \frac{M-3i}{2} \right] \right) (M'+1),$$

where  $[\cdot]$  is the integer part of a number.

**Example 5.** Consider the case  $M = 4, M' = 3$ . There are 16 points of  $F_{4,3}$  (see figure 4(b))

$$|F_{4,3}(G_2 \times A_1)| = \left( \left[ \frac{4}{3} \right] + 1 + \sum_{i=0}^{\lfloor \frac{4}{3} \rfloor} \left[ \frac{4-3i}{2} \right] \right) (3+1) = (1+1+(2+0)) \cdot 4 = 16.$$

Explicitly, we have the following sets of integers  $[s_1, s_2, s'_1]$  and the corresponding grid points in the  $\check{\omega}$ -basis,  $(\frac{s_1}{M}, \frac{s_2}{M}, \frac{s'_1}{M'})$ :

$$\begin{aligned} [0, 0, 0] &= (0, 0, 0), & [0, 0, 1] &= (0, 0, \frac{1}{3}), & [0, 0, 2] &= (0, 0, \frac{2}{3}), & [0, 0, 3] &= (0, 0, 1), \\ [0, 1, 0] &= (0, \frac{1}{4}, 0), & [0, 1, 1] &= (0, \frac{1}{4}, \frac{1}{3}), & [0, 1, 2] &= (0, \frac{1}{4}, \frac{2}{3}), & [0, 1, 3] &= (0, \frac{1}{4}, 1), \\ [1, 0, 0] &= (\frac{1}{4}, 0, 0), & [1, 0, 1] &= (\frac{1}{4}, 0, \frac{1}{3}), & [1, 0, 2] &= (\frac{1}{4}, 0, \frac{2}{3}), & [1, 0, 3] &= (\frac{1}{4}, 0, 1), \\ [2, 0, 0] &= (\frac{1}{2}, 0, 0), & [2, 0, 1] &= (\frac{1}{2}, 0, \frac{1}{3}), & [2, 0, 2] &= (\frac{1}{2}, 0, \frac{2}{3}), & [2, 0, 3] &= (\frac{1}{2}, 0, 1). \end{aligned}$$

Next, consider the case  $M = 3, M' = 2$ .  $F_{3,2}(G_2 \times A_1)$  consists of the following points:

$$\begin{aligned} [0, 0, 0] &= (0, 0, 0), & [0, 0, 1] &= (0, 0, \frac{1}{2}), & [0, 0, 2] &= (0, 0, 1), \\ [1, 0, 0] &= (\frac{1}{3}, 0, 0), & [1, 0, 1] &= (\frac{1}{3}, 0, \frac{1}{2}), & [1, 0, 2] &= (\frac{1}{3}, 0, 1), \\ [0, 1, 0] &= (0, \frac{1}{3}, 0), & [0, 1, 1] &= (0, \frac{1}{3}, \frac{1}{2}), & [0, 1, 2] &= (0, \frac{1}{3}, 1). \end{aligned}$$

In this case, none of the nine points of  $F_{3,2}(G_2 \times A_1)$  coincides with a point of  $F_{4,3}(G_2 \times A_1)$ . This is due to the fact that lattice densities  $M = 3$  and  $M' = 2$  do not correspondingly divide the densities  $M = 4$  and  $M' = 3$ .

The Weyl group orbit of the generic point  $a\omega_1 + b\omega_2 + c\omega_3, a, b, c > 0$ , consists of 24 points

$$\begin{aligned} W_{(a,b,c)}(G_2 \times A_1) &= \{ \pm(a, b, c), \pm(-a, 3a+b, c), \pm(a+b, -b, c), \\ &\pm(2a+b, -(3a+b), c), \pm(-(a+b), 3a+2b, c), \pm(-(2a+b), 3a+2b, c), \\ &\pm(a, b, -c), \pm(-a, 3a+b, -c), \pm(a+b, -b, -c), \pm(2a+b, -(3a+b), -c), \\ &\pm(-(a+b), 3a+2b, -c), \pm(-(2a+b), 3a+2b, -c) \}. \end{aligned}$$

Orbit sizes for arbitrary points are given by the relations

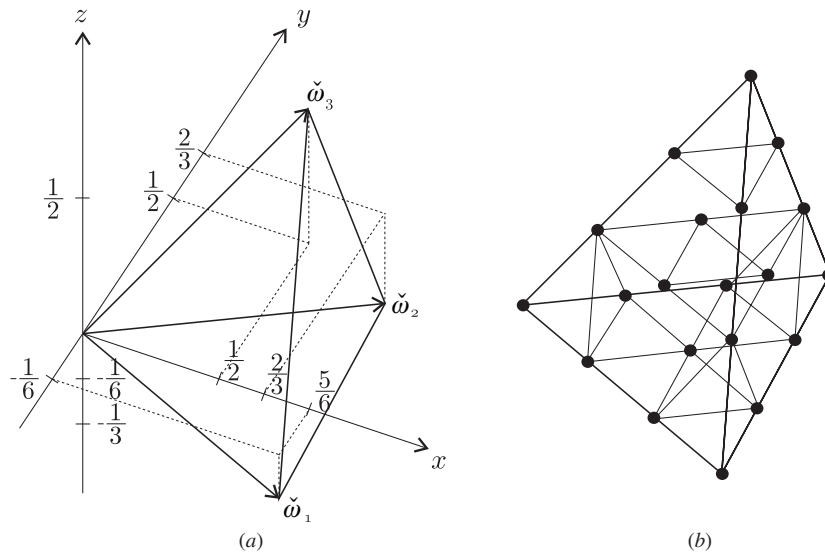
$$\begin{aligned} |W_{(a,b,c)}| &= 24, & |W_{(a,b,0)}| &= 12, & |W_{(a,0,c)}| &= 12, & |W_{(0,b,c)}| &= 12, \\ |W_{(a,0,0)}| &= 6, & |W_{(0,b,0)}| &= 6, & |W_{(0,0,c)}| &= 2, & |W_{(0,0,0)}| &= 1. \end{aligned}$$

### 3.5. The Lie algebra $A_3$

The Dynkin diagram and Cartan matrix with its inverse are the following,

$$\begin{array}{c} 1 & 1 & 1 \\ \circ & \circ & \circ \\ \alpha_1 & \alpha_2 & \alpha_3 \end{array} \quad C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad C^{-1} = \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

Hence, all simple roots of the same length equal to  $\sqrt{2}$ .



**Figure 5.** (a) The fundamental region  $F$  of the Lie algebra  $A_3$ ;  $x$ ,  $y$  and  $z$  indicate respectively the orthogonal directions of the orthonormal basis  $\{e_1, e_2, e_3\}$ , and the fundamental region of  $A_3$  is the tetrahedron; (b) the grid  $F_3(A_3)$ .

The bases of simple roots and fundamental weights are thus related by

$$\begin{aligned} \alpha_1 &= 2\omega_1 - \omega_2, & \omega_1 &= \frac{3}{4}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{4}\alpha_3, & \check{\alpha}_i &= \alpha_i, & i &\in \{1, 2, 3\}. \\ \alpha_2 &= -\omega_1 + 2\omega_2 - \omega_3, & \omega_2 &= \frac{1}{2}\alpha_1 + \alpha_2 + \frac{1}{2}\alpha_3, & \check{\omega}_i &= \omega_i, & i &\in \{1, 2, 3\}. \\ \alpha_3 &= -\omega_2 + 2\omega_3; & \omega_3 &= \frac{1}{4}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{3}{4}\alpha_3; \end{aligned}$$

Relative to the orthonormal basis, we have

$$\begin{aligned} \alpha_1 &= (1, -1, 0) = e_1 - e_2, & \omega_1 &= \left(\frac{5}{6}, -\frac{1}{6}, -\frac{1}{6}\right) = \frac{1}{6}(5e_1 - e_2 - e_3) = \check{\omega}_1, \\ \alpha_2 &= (0, 1, -1) = e_2 - e_3, & \omega_2 &= \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right) = \frac{1}{3}(2e_1 + 2e_2 - e_3) = \check{\omega}_2, \\ \alpha_3 &= \left(\frac{1}{3}, \frac{1}{3}, \frac{4}{3}\right) = \frac{1}{3}(e_1 + e_2 + 4e_3); & \omega_3 &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}(e_1 + e_2 + e_3) = \check{\omega}_3. \end{aligned}$$

The highest root  $\xi$  is given by the formula

$$\xi = \alpha_1 + \alpha_2 + \alpha_3.$$

The fundamental region is a tetrahedron (see figure 5(a)) with vertices

$$F(A_3) = \{0, \check{\omega}_1, \check{\omega}_2, \check{\omega}_3\}.$$

The volume of the fundamental region is given by the formula

$$|F(A_3)| = \frac{1}{6} \langle [\check{\omega}_1, \check{\omega}_2], \check{\omega}_3 \rangle = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \begin{vmatrix} 5 & -1 & -1 \\ 2 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{12},$$

where  $[\cdot, \cdot]$  denotes the vector product.

The grid  $F_M \subset F$  is fixed by the choice of one positive integer  $M$ . It consists of all the points

$$F_M(A_3) = \left\{ \frac{s_1}{M}\check{\omega}_1 + \frac{s_2}{M}\check{\omega}_2 + \frac{s_3}{M}\check{\omega}_3 \mid s_1 + s_2 + s_3 \leq M; s_1, s_2, s_3 \in \mathbb{Z}^{\geq 0} \right\}.$$

The number of points in the grid  $F_M$  equals

$$|F_M(A_3)| = \frac{1}{2} \sum_{i=0}^M (M+1-i)(M+2-i).$$

**Example 6.** Consider the case  $M = 3$ . There are 20 points of  $F_3(A_3)$  (see figure 5(b)). Explicitly, we have the following sets of integers  $[s_1, s_2, s_3]$  and the corresponding grid points in the  $\check{\omega}$ -basis  $(\frac{s_1}{M}, \frac{s_2}{M}, \frac{s_3}{M})$ :

$$\begin{aligned} [0, 0, 0] &= (0, 0, 0), & [0, 0, 3] &= (0, 0, 1), & [0, 3, 0] &= (0, 1, 0), & [3, 0, 0] &= (1, 0, 0), \\ [2, 0, 1] &= (\frac{2}{3}, 0, \frac{1}{3}), & [2, 1, 0] &= (\frac{2}{3}, \frac{1}{3}, 0), & [1, 0, 2] &= (\frac{1}{3}, 0, \frac{2}{3}), & [0, 1, 2] &= (0, \frac{1}{3}, \frac{2}{3}), \\ [1, 2, 0] &= (\frac{1}{3}, \frac{2}{3}, 0), & [0, 2, 1] &= (0, \frac{2}{3}, \frac{1}{3}), & [1, 0, 0] &= (\frac{1}{3}, 0, 0), & [0, 0, 1] &= (0, 0, \frac{1}{3}), \\ [0, 1, 0] &= (0, \frac{1}{3}, 0), & [1, 1, 0] &= (\frac{1}{3}, \frac{1}{3}, 0), & [1, 0, 1] &= (\frac{1}{3}, 0, \frac{1}{3}), & [0, 1, 1] &= (0, \frac{1}{3}, \frac{1}{3}), \\ [1, 1, 1] &= (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), & [2, 0, 0] &= (\frac{2}{3}, 0, 0), & [0, 0, 2] &= (0, 0, \frac{2}{3}), & [0, 2, 0] &= (0, \frac{2}{3}, 0). \end{aligned}$$

The Weyl group orbit of the generic point  $a\omega_1 + b\omega_2 + c\omega_3, a, b, c > 0$ , consists of 24 points

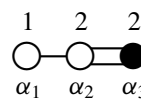
$$\begin{aligned} W_{(a,b,c)}(A_3) &= \{(a, b, c), (-a, a+b, c), (a+b, -b, b+c), \\ & (a, b+c, -c), (b, -(a+b), a+b+c), \\ & (-a, a+b+c, -c), (-(a+b), a, b+c), (a+b, c, -(b+c)), \\ & (a+b+c, -(b+c), c), (b, c, -(a+b+c)), (b+c, -(a+b+c), a+b), \\ & (-b, -a, a+b+c), (-(a+b), a+b+c, -(b+c)), (-(a+b+c), a, b), \\ & (a+b+c, -c, -b), (b+c, -c, -(a+b)), (-b, b+c-(a+b+c)), \\ & (c, -(a+b+c), a), (-(b+c), -a, (a+b)), (-(a+b+c), a+b, -b), \\ & (c, -(b+c), -a), (-(b+c), b, -(a+b)), (-c, -(a+b), a), (-c, -b, -a)\}. \end{aligned}$$

Orbit sizes for arbitrary points are given by the relations

$$\begin{aligned} |W_{(a,b,c)}| &= 24, & |W_{(a,b,0)}| &= 12, & |W_{(a,0,c)}| &= 12, & |W_{(0,b,c)}| &= 12, \\ |W_{(a,0,0)}| &= 4, & |W_{(0,b,0)}| &= 6, & |W_{(0,0,c)}| &= 4, & |W_{(0,0,0)}| &= 1. \end{aligned}$$

### 3.6. The Lie algebra $B_3$

The Dynkin diagram and Cartan matrix with its inverse are the following,



$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \quad C^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 4 \\ 1 & 2 & 3 \end{pmatrix}.$$

Hence  $\alpha_1$  is the shorter of the simple roots. The relative lengths of the simple roots are set as  $\langle \alpha_1, \alpha_1 \rangle = 1$  and  $\langle \alpha_2, \alpha_2 \rangle = \langle \alpha_3, \alpha_3 \rangle = 2$ .

The bases of simple roots and fundamental weights are thus related by

$$\begin{aligned} \alpha_1 &= 2\omega_1 - \omega_2, & \omega_1 &= \alpha_1 + \alpha_2 + \alpha_3, & \check{\alpha}_1 &= \alpha_1, & \check{\omega}_1 &= \omega_1, \\ \alpha_2 &= -\omega_1 + 2\omega_2 - 2\omega_3, & \omega_2 &= \alpha_1 + 2\alpha_2 + 2\alpha_3, & \check{\alpha}_2 &= \alpha_2, & \check{\omega}_2 &= \omega_2, \\ \alpha_3 &= -\omega_2 + 2\omega_3; & \omega_3 &= \frac{1}{2}\alpha_1 + \alpha_2 + \frac{3}{2}\alpha_3; & \check{\alpha}_3 &= 2\alpha_3; & \check{\omega}_3 &= 2\omega_3. \end{aligned}$$



Relative to the orthonormal basis, we have

$$\begin{aligned} \alpha_1 &= (1, -1, 0) = e_1 - e_2, & \omega_1 &= (1, 0, 0) = e_1, & \check{\omega}_1 &= (1, 0, 0) = e_1, \\ \alpha_2 &= (0, 1, -1) = e_2 - e_3, & \omega_2 &= (1, 1, 0) = e_1 + e_2, & \check{\omega}_2 &= (1, 1, 0) = e_1 + e_2, \\ \alpha_3 &= (0, 0, 1) = e_3; & \omega_3 &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}(e_1 + e_2 + e_3); & \check{\omega}_3 &= (1, 1, 1) = e_1 + e_2 + e_3. \end{aligned}$$

The highest root  $\xi$  is given by the formula

$$\xi = \alpha_1 + 2\alpha_2 + 2\alpha_3.$$

The fundamental region is a tetrahedron (see figure 1(a)) with vertices

$$F(B_3) = \left\{0, \check{\omega}_1, \frac{1}{2}\check{\omega}_2, \frac{1}{2}\check{\omega}_3\right\}.$$

The volume of the fundamental region is given by the formula

$$|F(B_3)| = \frac{1}{6} \left| \left[ \check{\omega}_1, \frac{1}{2}\check{\omega}_2, \frac{1}{2}\check{\omega}_3 \right] \right| = \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{1}{2} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{24},$$

where  $[\cdot, \cdot]$  denotes the vector product.

The grid  $F_M \subset F$  is fixed by the choice of an integer  $M$ . It consists of all the points

$$F_M(B_3) = \left\{ \frac{s_1}{M}\check{\omega}_1 + \frac{s_2}{M}\check{\omega}_2 + \frac{s_3}{M}\check{\omega}_3 \mid s_1 + 2s_2 + 2s_3 \leq M; s_1, s_2, s_3 \in \mathbb{Z}^{\geq 0} \right\}.$$

The number of points in the grid  $F_M$  equals

$$|F_M(B_3)| = \left( \left[ \frac{M}{2} \right] + 1 \right) \left( \left[ \frac{M}{2} \right] \left[ \frac{M+1}{2} \right] + M + 1 - \frac{M+2}{2} \left[ \frac{M}{2} \right] \right) + \sum_{i=0}^{\left[ \frac{M}{2} \right]} i^2.$$

where  $[\cdot]$  is the integer part of a number.

**Example 7.** Consider the case  $M = 4$ , see figure 6(b). There are 14 points of  $F_4$

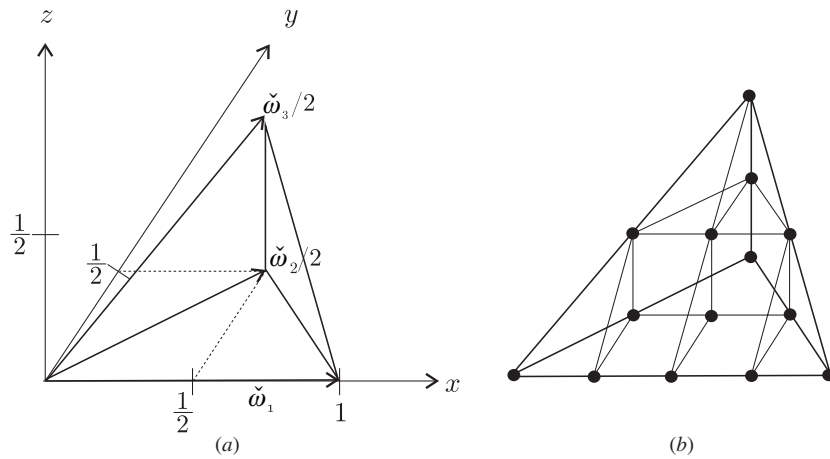
$$\begin{aligned} |F_4(B_3)| &= \left( \left[ \frac{4}{2} \right] + 1 \right) \left( \left[ \frac{4}{2} \right] \left[ \frac{4+1}{2} \right] + 4 + 1 - \frac{4+2}{2} \left[ \frac{4}{2} \right] \right) + \sum_{i=0}^{\left[ \frac{4}{2} \right]} i^2 \\ &= 3(2 \cdot 2 + 4 + 1 - 3 \cdot 2) + 0 + 1 + 4 = 14. \end{aligned}$$

Explicitly, we have the following sets of integers  $[s_1, s_2, s_3]$  and the corresponding grid points in the  $\check{\omega}$ -basis,  $\left(\frac{s_1}{M}, \frac{s_2}{M}, \frac{s_3}{M}\right)$ :

$$\begin{aligned} [0, 0, 0] &= (0, 0, 0), & [0, 0, 1] &= \left(0, 0, \frac{1}{4}\right), & [0, 0, 2] &= \left(0, 0, \frac{1}{2}\right), & [1, 0, 1] &= \left(\frac{1}{4}, 0, \frac{1}{4}\right), \\ [0, 2, 0] &= \left(0, \frac{1}{2}, 0\right), & [0, 1, 1] &= \left(0, \frac{1}{4}, \frac{1}{4}\right), & [0, 1, 2] &= \left(0, \frac{1}{4}, \frac{1}{2}\right), & [1, 0, 2] &= \left(\frac{1}{4}, 0, \frac{1}{2}\right), \\ [0, 0, 4] &= (0, 0, 1), & [0, 0, 3] &= \left(0, 0, \frac{3}{4}\right), & [1, 1, 0] &= \left(\frac{1}{4}, \frac{1}{4}, 0\right), & [2, 0, 0] &= \left(\frac{1}{2}, 0, 0\right), \\ [0, 1, 0] &= \left(0, \frac{1}{4}, 0\right), & [1, 0, 0] &= \left(\frac{1}{4}, 0, 0\right), \end{aligned}$$

The Weyl group orbit of the generic point  $a\omega_1 + b\omega_2 + c\omega_3$ ,  $a, b, c > 0$ , consists of 48 points

$$\begin{aligned} W_{(a,b,c)}(B_3) &= \{\pm(a, b, c), \pm(-a, a+b, c), \pm(a+b, -b, 2b+c), \pm(a, b+c, -c), \\ &\quad \pm(b, -(a+b), 2a+2b+c), \pm(-a, a+b+c, -c), \pm(-(a+b), a, 2b+c), \\ &\quad \pm(a+b, b+c, -(2b+c)), \pm(a+b+c, -(b+c), 2b+c), \\ &\quad \pm(b, a+b+c, -(2a+2b+c)), \pm(b+c, -(a+b+c), 2a+2b+c), \end{aligned}$$



**Figure 6.** (a) The fundamental region  $F$  of the Lie algebra  $B_3$ ;  $x$ ,  $y$  and  $z$  indicate respectively the orthogonal directions of the orthonormal basis  $\{e_1, e_2, e_3\}$  and the face defined by vertices  $\{0, \check{\omega}_1, \frac{1}{2}\check{\omega}_2\}$  lies in the plane  $z = 0$ ; (b) the grid  $F_4(B_3)$ .

$$\begin{aligned} & \pm(-b, -a, 2a + 2b + c), \pm(-(a + b), a + 2b + c, -(2b + c)), \\ & \pm(a + 2b + c, -(b + c), c), \pm(-(a + b + c), a, 2b + c), \\ & \pm(a + b + c, b, -(2b + c)), \pm(a + 2b + c, -(a + b + c), c), \\ & \pm(b + c, a + b, -(2a + 2b + c)), \pm(-b, a + 2b + c, -(2a + 2b + c)), \\ & \pm(b + c, -(a + 2b + c), 2a + 2b + c), \pm(-(a + 2b + c), a + b, c), \\ & \pm(a + 2b + c, -b, -c), \pm(-(b + c), -a, 2a + 2b + c), \\ & \pm(-(a + b + c), a + 2b + c, -(2b + c)) \}. \end{aligned}$$

Orbit sizes for arbitrary points are given by the relations

$$\begin{aligned} |W_{(a,b,c)}| &= 48, & |W_{(a,b,0)}| &= 24, & |W_{(a,0,c)}| &= 24, & |W_{(0,b,c)}| &= 24, \\ |W_{(a,0,0)}| &= 6, & |W_{(0,b,0)}| &= 12, & |W_{(0,0,c)}| &= 8, & |W_{(0,0,0)}| &= 1. \end{aligned}$$

### 3.7. The Lie algebra $C_3$

The Dynkin diagram and Cartan matrix with its inverse are the following,

$$\begin{array}{c} \bullet \quad \bullet \quad \circ \\ \alpha_1 \quad \alpha_2 \quad \alpha_3 \end{array} \quad C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}, \quad C^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 2 \\ 2 & 4 & 3 \end{pmatrix}.$$

Hence  $\alpha_3$  is the longer of the simple roots. The relative lengths of the simple roots are set as  $\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = 1$  and  $\langle \alpha_3, \alpha_3 \rangle = 2$ .

The bases of simple roots and fundamental weights are thus related by

$$\begin{aligned} \alpha_1 &= 2\omega_1 - \omega_2, & \omega_1 &= \alpha_1 + \alpha_2 + \frac{1}{2}\alpha_3, & \check{\alpha}_1 &= 2\alpha_1, & \check{\omega}_1 &= 2\omega_1, \\ \alpha_2 &= -\omega_1 + 2\omega_2 - \omega_3, & \omega_2 &= \alpha_1 + 2\alpha_2 + \alpha_3, & \check{\alpha}_2 &= 2\alpha_2, & \check{\omega}_2 &= 2\omega_2, \\ \alpha_3 &= -2\omega_1 + 2\omega_3; & \omega_3 &= \alpha_1 + 2\alpha_2 + \frac{3}{2}\alpha_3; & \check{\alpha}_3 &= \alpha_3; & \check{\omega}_3 &= \omega_3. \end{aligned}$$

In the orthonormal basis these bases have the form

$$\begin{aligned} \alpha_1 &= \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) = \frac{1}{\sqrt{2}}(e_1 - e_2), & \omega_1 &= \left(\frac{1}{\sqrt{2}}, 0, 0\right) = \frac{1}{\sqrt{2}}e_1, & \check{\omega}_1 &= (\sqrt{2}, 0, 0), \\ \alpha_2 &= \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(e_2 - e_3), & \omega_2 &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \frac{1}{\sqrt{2}}(e_1 + e_2), \\ & & \check{\omega}_2 &= (\sqrt{2}, \sqrt{2}, 0), \\ \alpha_3 &= (0, 0, \sqrt{2}) = \sqrt{2}e_3; & \omega_3 &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(e_1 + e_2 + e_3); \\ & & \check{\omega}_3 &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right). \end{aligned}$$

The highest root  $\xi$  is given by the formula

$$\xi = 2\alpha_1 + 2\alpha_2 + \alpha_3.$$

The fundamental region is a tetrahedron (see figure 7(a)) with vertices

$$F(C_3) = \left\{0, \frac{1}{2}\check{\omega}_1, \frac{1}{2}\check{\omega}_2, \check{\omega}_3\right\}.$$

The volume of the fundamental region is given by the formula

$$|F(C_3)| = \frac{1}{6} \left| \left[ \frac{1}{2}\check{\omega}_1, \frac{1}{2}\check{\omega}_2, \check{\omega}_3 \right] \right| = \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \frac{\sqrt{2}}{24},$$

where  $[\cdot, \cdot]$  is the vector product.

The grid  $F_M \subset F$  is fixed by the choice of the integer  $M$ . It consists of all the points

$$F_M(C_3) = \left\{ \frac{s_1}{M}\check{\omega}_1 + \frac{s_2}{M}\check{\omega}_2 + \frac{s_3}{M}\check{\omega}_3 \mid 2s_1 + 2s_2 + s_3 \leq M; s_1, s_2, s_3 \in \mathbb{Z}^{\geq 0} \right\}.$$

The number of points in the grid  $F_M$  equals

$$|F_M(C_3)| = \left( \left[ \frac{M}{2} \right] + 1 \right) \left( \left[ \frac{M}{2} \right] \left[ \frac{M+1}{2} \right] + M + 1 - \frac{M+2}{2} \left[ \frac{M}{2} \right] \right) + \sum_{i=0}^{\left[ \frac{M}{2} \right]} i^2,$$

where  $[\cdot]$  is the integer part of a number.

**Example 8.** Consider the case  $M = 4$ . There are 14 points of  $F_4(C_3)$ , see figure 7(b).

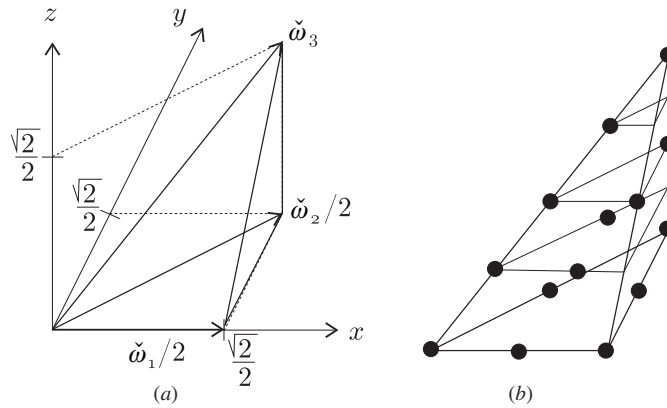
$$|F_4(C_3)| = \left( \left[ \frac{4}{2} \right] + 1 \right) \left( \left[ \frac{4}{2} \right] \left[ \frac{4+1}{2} \right] + 4 + 1 - \frac{4+2}{2} \left[ \frac{4}{2} \right] \right) + \sum_{i=0}^{\left[ \frac{4}{2} \right]} i^2 = 14.$$

Explicitly, we have the following sets of integers  $[s_1, s_2, s_3]$  and the corresponding grid points in the  $\check{\omega}$ -basis,  $\left(\frac{s_1}{M}, \frac{s_2}{M}, \frac{s_3}{M}\right)$ :

$$\begin{aligned} [0, 0, 0] &= (0, 0, 0), & [0, 0, 1] &= \left(0, 0, \frac{1}{4}\right), & [0, 0, 2] &= \left(0, 0, \frac{1}{2}\right), & [0, 0, 3] &= \left(0, 0, \frac{3}{4}\right), \\ [0, 1, 0] &= \left(0, \frac{1}{4}, 0\right), & [0, 1, 1] &= \left(0, \frac{1}{4}, \frac{1}{4}\right), & [0, 0, 4] &= (0, 0, 1), & [2, 0, 0] &= \left(\frac{1}{2}, 0, 0\right), \\ [1, 0, 0] &= \left(\frac{1}{4}, 0, 0\right), & [1, 0, 1] &= \left(1, 0, \frac{1}{3}\right), & [1, 0, 2] &= \left(1, 0, \frac{2}{3}\right), & [1, 1, 2] &= \left(1, 1, \frac{2}{3}\right), \\ [1, 1, 0] &= (1, 1, 0), & [1, 1, 1] &= \left(1, 1, \frac{1}{3}\right), \end{aligned}$$

The Weyl group orbit of the generic point  $a\omega_1 + b\omega_2 + c\omega_3$ ,  $a, b, c > 0$ , consists of 48 points

$$\begin{aligned} W_{(a,b,c)}(C_3) &= \{\pm(a, b, c), \pm(-a, a+b, c), \pm(a+b, -b, b+c), \pm(a, b+2c, -c), \\ &\pm(b, -(a+b), a+b+c), \pm(-a, a+b+2c, -c), \pm(-(a+b), a, b+c)\}, \end{aligned}$$



**Figure 7.** (a) The fundamental region  $F$  of the Lie algebra  $C_3$ ;  $x$ ,  $y$  and  $z$  indicate respectively the orthogonal directions of the orthonormal basis  $\{e_1, e_2, e_3\}$  and the face defined by vertices  $\{0, \frac{1}{2}\check{\omega}_1, \frac{1}{2}\check{\omega}_2\}$  lies in the plane  $z = 0$ ; (b) the grid  $F_4(C_3)$ .

$$\begin{aligned} & \pm(a + b, b + 2c, -(b + c)), \pm(a + b + 2c, -(b + 2c), b + c), \\ & \pm(b, a + b + 2c, -(a + b + c)), \pm(b + 2c, -(a + b + 2c), a + b + c), \\ & \pm(-b, -a, a + b + c), \pm(-(a + b), a + 2b + 2c, -(b + c)), \\ & \pm(a + 2b + 2c, -(b + 2c), c), \pm(-(a + b + 2c), a, b + c), \\ & \pm(a + b + 2c, b, -(b + c)), \pm(a + 2b + 2c, -(a + b + 2c), c), \\ & \pm(b + 2c, a + b, -(a + b + c)), \pm(-b, a + 2b + 2c, -(a + b + c)), \\ & \pm(b + 2c, -(a + 2b + 2c), a + b + c), \pm(-(a + 2b + 2c), a + b, c), \\ & \pm(a + 2b + 2c, -b, -c), \pm(-(b + 2c), -a, a + b + c), \\ & \pm(-(a + b + 2c), a + 2b + 2c, -(b + c)) \}. \end{aligned}$$

Orbit sizes for arbitrary points are given by the relations

$$\begin{aligned} |W_{(a,b,c)}| &= 48, & |W_{(a,b,0)}| &= 24, & |W_{(a,0,c)}| &= 24, & |W_{(0,b,c)}| &= 24, \\ |W_{(a,0,0)}| &= 6, & |W_{(0,b,0)}| &= 12, & |W_{(0,0,c)}| &= 8, & |W_{(0,0,0)}| &= 1. \end{aligned}$$

#### 4. Orbit functions

In this section, we define what we mean by  $C$ -,  $S$ - and  $E$ -functions, specified by a given point  $\lambda \in \mathbb{Z}^n$ . We also show some of the properties inherent to those functions. Namely, the following properties are of interest, their pairwise orthogonality (using the appropriate scalar product, an integral over the fundamental region), their discrete orthogonality (again, using a properly defined scalar product, a sum over the discrete grid), their product can be represented as a sum, and they are eigenfunctions of the Laplace operator.

##### 4.1. Definitions, symmetries and general properties

We start with the  $C$ -functions. The  $C$ -function  $C_\lambda(x)$ ,  $\lambda \in P^+$  is defined as

$$C_\lambda(x) := \sum_{\mu \in W_\lambda} e^{2\pi i(\mu, x)}, \quad x \in \mathbb{R}^n, \tag{7}$$

where  $W_\lambda$  is the Weyl group orbit generated from  $\lambda$ .

If in (7) we restrict ourselves to the orbit of the even subgroup  $W_{e\lambda}$ , then we define  $E$ -function  $E_\lambda(x)$ ,  $\lambda \in P_e$

$$E_\lambda(x) := \sum_{\mu \in W_{e\lambda}} e^{2\pi i(\mu, x)}, \quad x \in \mathbb{R}^n. \tag{8}$$

The definition of an  $S$ -function  $S_\lambda(x)$ ,  $\lambda \in P^{++}$  is almost identical, but the sign of each summand is determined by the number of reflections  $p(\mu)$  necessary to obtain  $\mu$  from  $\lambda$

$$S_\lambda(x) := \sum_{\mu \in W_\lambda} (-1)^{p(\mu)} e^{2\pi i(\mu, x)} \quad x \in \mathbb{R}^n. \tag{9}$$

Of course the same  $\mu$  can be obtained by different successions of reflections, but all routes from  $\lambda$  to  $\mu$  will have a length of the same parity, and thus the salient detail given by  $p(\mu)$ , in the context of an  $S$ -function, is meaningful and unchanging.

For different families of orbit functions, the  $\lambda$  (represented in the  $\omega$ -basis) are taken from different sets, namely

$$\begin{aligned} \lambda \in \{\mathbb{Z}^{\geq 0}\omega_1 + \mathbb{Z}^{\geq 0}\omega_2 + \mathbb{Z}^{\geq 0}\omega_3\}, & \quad \text{for } C\text{-functions;} \\ \lambda \in \{\mathbb{Z}^{\geq 0}\omega_1 + \mathbb{Z}^{\geq 0}\omega_2 + \mathbb{Z}^{\geq 0}\omega_3\} \cup r_i\{\mathbb{Z}^{> 0}\omega_1 + \mathbb{Z}^{> 0}\omega_2 + \mathbb{Z}^{> 0}\omega_3\}, r_i \in W & \quad \text{for } E\text{-functions;} \\ \lambda \in \{\mathbb{Z}^{> 0}\omega_1 + \mathbb{Z}^{> 0}\omega_2 + \mathbb{Z}^{> 0}\omega_3\}, & \quad \text{for } S\text{-functions.} \end{aligned}$$

In particular, this implies that, for  $S$ -functions, the number of summands always equals the size of the Weyl group.

In the case of  $x \in F_M$  (the coordinates of  $x$  are rational), the  $C$ -,  $S$ - and  $E$ -functions are formed by roots of unity. Therefore, there can only be a finite number of possible orbit functions that can take distinct values on the points of  $F_M$ , these functions are orthogonal on the grid. The number of pairwise orthogonal orbit functions on  $F_M$  coincides with the size of  $F_M$ , including the boundary in the case of  $C$ - and  $E$ -functions and excluding the boundary in the case of,  $S$ -functions.

Note that in the one-dimensional case,  $C$ -,  $S$ - and  $E$ -functions are respectively a cosine, a sine and an exponential functions up to the constant.

All three families of orbit functions are based on semisimple Lie groups of finite order, the number of variables coincides with the rank of the corresponding Lie algebra.

In general,  $C$ -,  $S$ - and  $E$ -functions are the finite sums of exponential functions, therefore they are continuous and have continuous derivatives of all orders in  $\mathbb{R}^n$ .

The  $S$ -functions are antisymmetric with respect to  $(n - 1)$ -dimensional boundary of  $F$ . Hence they are zero on the boundary of  $F$ . The  $C$ -functions are symmetric with respect to  $(n - 1)$ -dimensional boundary of  $F$ . Their normal derivative at the boundary is equal to zero (because the normal derivative of a  $C$ -function is an  $S$ -function). A number of other properties of orbit functions are presented in [11–13], see also forthcoming papers [6–8].

#### 4.2. Calculation of scalar products

Here we present the rules necessary for the calculation of the scalar products of the vectors given in the different bases.

Let vectors  $u = (u_1, u_2, \dots, u_n) = u_1\omega_1 + u_2\omega_2 + \dots + u_n\omega_n$  and  $v = v_1\omega_1 + v_2\omega_2 + \dots + v_n\omega_n$  are represented in the  $\omega$ -basis, then their scalar product is calculated as the matrix product

$$\langle u, v \rangle = u \tilde{C} v^t = \sum_{i,j=1}^n u_i v_j \langle \omega_i, \omega_j \rangle = \sum_{i,j=1}^n u_i v_j \frac{\langle \alpha_j, \alpha_j \rangle}{2} C_{i,j}^{-1},$$

here  $\tilde{C}_{i,j} = \frac{\langle \alpha_j, \alpha_j \rangle}{2} C_{i,j}^{-1}$  and  $C$  is the Cartan matrix.

If vectors  $x = (x_1, x_2, \dots, x_n) = x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n$  and  $y = y_1\alpha_1 + y_2\alpha_2 + \dots + y_n\alpha_n$  are represented in the  $\alpha$ -basis, then their scalar product is calculated as the matrix product

$$\langle x, y \rangle = x \hat{C} y^t = \sum_{i,j=1}^n x_i y_j \langle \alpha_i, \alpha_j \rangle = \sum_{i,j=1}^n x_i y_j \frac{\langle \alpha_j, \alpha_j \rangle}{2} C_{i,j},$$

here  $\hat{C}_{i,j} = \frac{\langle \alpha_j, \alpha_j \rangle}{2} C_{i,j}$  and  $C$  is the Cartan matrix.

Square lengths of the simple roots  $\langle \alpha_j, \alpha_j \rangle$  are always indicated in the Dynkin diagrams of the semisimple Lie groups, see e.g. [8, 10].

It is also useful to recall that

$$\langle \alpha_i, \check{\omega}_j \rangle = \langle \check{\alpha}_i, \omega_j \rangle = \langle e_i, e_j \rangle = \delta_{i,j}, \quad \text{where } \delta_{i,j} \text{ is the Kronecker delta.}$$

#### 4.3. $C$ -, $S$ - and $E$ -functions as eigenfunctions of the Laplace operator

Consider the functions  $C_\lambda(x)$ ,  $E_\lambda(x)$  and  $S_\lambda(x)$  and suppose that the continuous variable  $x$  is given relative to the orthogonal basis. In the case of Lie algebra  $A_n$  we use orthogonal coordinates  $x_1, x_2, \dots, x_{n+1}$  and coordinates  $x_1, x_2, \dots, x_n$  for  $B_n, C_n$  and  $D_n$  (the orthogonal bases for these algebras are well known and can be found e.g. in [11]).

The Laplace operator in orthogonal coordinates has the form

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_k^2}, \quad \text{where } k = n \text{ (or } k = n + 1 \text{ for } A_n).$$

For the algebras  $A_n, B_n, C_n$  and  $D_n$ , the Laplace operator gives the same eigenvalues on every exponential function summand of an orbit function with eigenvalue  $-4\pi \langle \lambda, \lambda \rangle$ .

Hence, the functions  $C_\lambda(x)$ ,  $E_\lambda(x)$  and  $S_\lambda(x)$  are eigenfunctions of the Laplace operator:

$$\Delta \begin{pmatrix} C_\lambda(x) \\ E_\lambda(x) \\ S_\lambda(x) \end{pmatrix} = -4\pi^2 \langle \lambda, \lambda \rangle \begin{pmatrix} C_\lambda(x) \\ E_\lambda(x) \\ S_\lambda(x) \end{pmatrix}.$$

Now we consider the continuous variable  $x$  given relative to the  $\omega$ -basis. Let  $\Delta$  denote the Laplace operator, where the differentiation  $\partial_{x_i}$  is made with respect to the direction given by  $\omega_i$ .

$$\Delta = \sum_{i,j=1}^n \frac{C_{ij}}{\langle \alpha_i, \alpha_i \rangle} \partial_{x_i} \partial_{x_j}, \quad \text{where } C \text{ is the Cartan matrix.}$$

It is known in Lie theory that the matrix of scalar products of the simple roots is positive definite, moreover our definition makes matrix  $\frac{C_{ij}}{\langle \alpha_i, \alpha_i \rangle}$  symmetric, hence it can be diagonalized and the Laplace operator could be transformed to the sum of second derivatives by an appropriate change of variables.

Thereby, using the results of section 3 we can write the explicit forms of the Laplace operators given in the  $\omega$ -basis for all semisimple Lie algebras of rank 3

$$\Delta = \begin{cases} \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2, & \text{for } A_1 \times A_1 \times A_1; \\ \partial_{x_1}^2 - \partial_{x_1} \partial_{x_2} + \partial_{x_2}^2 + \partial_{x_3}^2, & \text{for } A_2 \times A_1; \\ 2\partial_{x_1}^2 - 2\partial_{x_1} \partial_{x_2} + \partial_{x_2}^2 + \partial_{x_3}^2, & \text{for } C_2 \times A_1; \\ 3\partial_{x_1}^2 - 5\partial_{x_1} \partial_{x_2} + \partial_{x_2}^2 + \partial_{x_3}^2, & \text{for } G_2 \times A_1; \\ \partial_{x_1}^2 - \partial_{x_1} \partial_{x_2} + \partial_{x_2}^2 - \partial_{x_2} \partial_{x_3} + \partial_{x_3}^2, & \text{for } A_3; \\ \partial_{x_1}^2 - \partial_{x_1} \partial_{x_2} + \partial_{x_2}^2 - 2\partial_{x_2} \partial_{x_3} + 2\partial_{x_3}^2, & \text{for } B_3; \\ 2\partial_{x_1}^2 - 2\partial_{x_1} \partial_{x_2} + 2\partial_{x_2}^2 - 2\partial_{x_2} \partial_{x_3} + \partial_{x_3}^2, & \text{for } C_3. \end{cases}$$

4.4. *Continuous orthogonality*

For any two squared integrable functions  $\phi(x)$  and  $\psi(x)$  defined on the fundamental region  $\tilde{F}$ , we define a continuous scalar product

$$\langle \phi(x), \psi(x) \rangle := \int_F \phi(x) \overline{\psi(x)} dx. \tag{10}$$

Here, integration is carried out with respect to the Euclidean measure, the bar means complex conjugation and  $x \in F$ , where  $F$  is the fundamental region of either  $W$  or  $W_e$ .

Any pair of orbit functions from the same family is orthogonal on the corresponding fundamental region with respect to the introduced scalar product (10), namely

$$\langle C_\lambda(x), C_{\lambda'}(x) \rangle = |W_\lambda| \cdot |F| \cdot \delta_{\lambda\lambda'}, \tag{11}$$

$$\langle S_\lambda(x), S_{\lambda'}(x) \rangle = |W| \cdot |F| \cdot \delta_{\lambda\lambda'}, \tag{12}$$

$$\langle E_\lambda(x), E_{\lambda'}(x) \rangle = |W_{e\lambda}| \cdot |F_e| \cdot \delta_{\lambda\lambda'}, \tag{13}$$

where  $\delta_{\lambda\lambda'}$  is the Kronecker delta,  $|W|$  is the size of Weyl group,  $|W_\lambda|$  and  $|W_{e\lambda}|$  are the sizes of Weyl group orbits, and  $|F|$  and  $|F_e|$  are volumes of fundamental regions. All of the necessary information for each semisimple Lie algebra of rank 3 can be found in section 3. In particular,  $|F_e| = 2|F|$  and  $|W_{e\lambda}|$  is defined by formula (5).

Proof of the relations (11)–(13) follows from the orthogonality of the usual exponential functions and from the fact that a given weight  $\mu \in P$  belongs to precisely one orbit function.

The families of  $C$ -,  $S$ - and  $E$ -functions are complete on the fundamental domain. The completeness of these systems is directly follows from the completeness of the system of exponential functions, i.e. there does not exist a function  $\phi(x)$  in the considered system of function such that  $\langle \phi(x), \phi(x) \rangle > 0$  and at the same time  $\langle \phi(x), \psi(x) \rangle = 0$  for all functions  $\psi(x)$  form the same system, see [11–13] for details.

Therefore each family of orbit functions forms an orthogonal basis in the Hilbert space of squared integrable functions  $\mathcal{L}^2(F)$ . Hence functions given on  $F$  can be expanded in terms of linear combinations of  $C$ -,  $S$ - or  $E$ -functions.

4.5. *Discrete orthogonality*

Let us denote the discrete grid of the fundamental region as  $F_M$  in the general case, even though in some cases it is determined by more than one positive integer

$$F_M = \begin{cases} F_M & \text{for } C\text{-functions,} \\ F_M \setminus \partial F & \text{for } S\text{-functions,} \\ F_{eM} & \text{for } E\text{-functions.} \end{cases}$$

A discrete scalar product of two functions  $\phi(x)$  and  $\psi(x)$  given on  $F_M$  (including  $C$ -,  $S$ - and  $E$ -functions) is dependent on this grid and defined by the bilinear form

$$\langle \phi(x), \psi(x) \rangle_M = \sum_{i=1}^{|F_M|} \varepsilon(x_i) \phi(x_i) \overline{\psi(x_i)}, \quad x_i \in F_M. \tag{14}$$

Here  $\varepsilon(x_i)$  is the number of points conjugate to  $x_i$  under the action of Weyl group  $W$  on the maximal torus of the Lie group (or even subgroup  $W_e \subset W$  in the case of  $E$ -functions). For  $S$ -functions the values of  $\varepsilon(x_i)$  are equal to  $|W|$ , for  $C$ -functions these values are given in appendix A.

Again, as in the continuous case, the  $C$ -,  $S$ - and  $E$ -functions are pairwise orthogonal, i.e.

$$\langle C_\lambda(x), C_{\lambda'}(x) \rangle_M = \sum_{i=1, \overline{N}} \varepsilon(x_i) C_\lambda(x_i) \overline{C_{\lambda'}(x_i)} = |W_\lambda| \cdot |A_M| \cdot \delta_{\lambda\lambda'}, \quad (15)$$

$$\langle S_\lambda(x), S_{\lambda'}(x) \rangle_M = |W| \sum_{i=1, \overline{N}} S_\lambda(x_i) \overline{S_{\lambda'}(x_i)} = |W| \cdot |A_M| \cdot \delta_{\lambda\lambda'}, \quad (16)$$

$$\langle E_\lambda(x), E_{\lambda'}(x) \rangle_M = \sum_{i=1, \overline{N}} \varepsilon(x_i) \phi_\lambda(x_i) \overline{\phi_{\lambda'}(x_i)} = |W_{e\lambda}| \cdot |A_M| \cdot \delta_{\lambda\lambda'}. \quad (17)$$

Here  $A_M$  denotes  $W$ -invariant Abelian subgroup of the grid points on the maximal torus  $\mathbb{T}$  of the simple compact group corresponding to  $W$

$$|A_M| = \sum_{i=1}^{|F_M|} \varepsilon(x_i), \quad x_i \in F_M.$$

Proof of the orthogonality relations can be found in [21]. All necessary data for the computation of the coefficients and discrete scalar products (i.e.  $|F_M|$  and  $\varepsilon(x_i)$ ) is given in section 3 and tables 1–3 of appendix A.

### 5. $C$ -, $S$ - and $E$ -transforms

For different fixed  $m \in R^n$  the set of exponential functions  $\{e^{2\pi i(m,x)}, x \in R^n\}$  determines continuous and discrete Fourier transforms on  $\mathbb{R}^n$ . In much the same way, the orbit functions (which are a symmetrized version of exponential functions defined in section 4) determine an analogue of the Fourier transform.

In this section, we introduce the essentials of the continuous and discrete  $C$ -,  $S$ - and  $E$ -transforms. The discrete transform can be used for the continuous interpolation of values of a function  $f(x)$  between its given values on a grid  $F_M$ .

#### 5.1. Continuous transforms

Each continuous function on the fundamental region with continuous derivatives can be expanded as the sum of  $C$ -,  $S$ - or  $E$ -functions. Let  $f(x)$  be a function defined on  $F$  (or  $F_e$  for  $E$ -functions), then it may be written that

$$f(x) = \sum_{\lambda \in P^+} c_\lambda C_\lambda(x), \quad c_\lambda = |W_\lambda|^{-1} |F|^{-1} \langle f(x), C_\lambda(x) \rangle; \quad (18)$$

$$f(x) = \sum_{\lambda \in P^{++}} c_\lambda S_\lambda(x), \quad c_\lambda = |W|^{-1} |F|^{-1} \langle f(x), S_\lambda(x) \rangle; \quad (19)$$

$$f(x) = \sum_{\lambda \in P_e} c_\lambda E_\lambda(x), \quad c_\lambda = |W_{e\lambda}|^{-1} |F_e|^{-1} \langle f(x), E_\lambda(x) \rangle. \quad (20)$$

Here  $\langle \cdot, \cdot \rangle$  denotes the continuous scalar product of (10). Direct and inverse  $C$ -,  $S$ - and  $E$ -transforms of the function  $f(x)$  are in (18), (19) and (20) respectively.

#### 5.2. Discrete transforms

Let  $\Lambda_M \in P$  be the maximal set of points, such that for any two  $\lambda, \lambda' \in \Lambda_M$  the condition of discrete orthogonality holds for any of the families of orbit functions in ((15), (16) or (17)).



Then we have the following discrete transforms for the function  $f(x)$ :

$$f(x) = \sum_{\lambda \in \Lambda_M} b_\lambda C_\lambda(x), \quad x \in F_M, \quad b_\lambda = \frac{\langle f, C_\lambda \rangle_M}{\langle C_\lambda, C_\lambda \rangle_M}; \quad (21)$$

$$f(x) = \sum_{\lambda \in \Lambda_M} b_\lambda S_\lambda(x), \quad x \in F_M, \quad b_\lambda = \frac{\langle f, S_\lambda \rangle_M}{\langle S_\lambda, S_\lambda \rangle_M}; \quad (22)$$

$$f(x) = \sum_{\lambda \in \Lambda_M} b_\lambda E_\lambda(x), \quad x \in F_{eM}, \quad b_\lambda = \frac{\langle f, E_\lambda \rangle_M}{\langle E_\lambda, E_\lambda \rangle_M}. \quad (23)$$

Here  $\langle \cdot, \cdot \rangle_M$  denotes the discrete scalar product given of (14).

### 5.3. Continuous extensions

Once the coefficients  $b_\lambda$  of the expansions (21)–(23) are calculated, discrete variables  $x_i$  in  $F_M$  may be replaced by continuous variables  $x$  in  $F$

$$\begin{aligned} f_{\text{cont}}(x) &:= \sum_{\lambda \in \Lambda_M} b_\lambda C_\lambda(x), & x \in F; \\ f_{\text{cont}}(x) &:= \sum_{\lambda \in \Lambda_M} b_\lambda S_\lambda(x), & x \in F; \\ f_{\text{cont}}(x) &:= \sum_{\lambda \in \Lambda_M} b_\lambda E_\lambda(x), & x \in F_e. \end{aligned}$$

The function  $f_{\text{cont}}(x)$  smoothly interpolates the values of  $f(x_i)$ ,  $i = 1, 2, \dots, |F_M|$ . At the points  $x_i$ , we have the equality  $f_{\text{cont}}(x_i) = f(x_i)$ .

**Remark 4.** If we calculate more than one discrete transform on the same grid  $F_M$  and use the same set of orbit functions  $\phi_\lambda(x)$ ,  $\lambda \in \Lambda_M$ , then it is reasonable to pre-compute and save the matrix

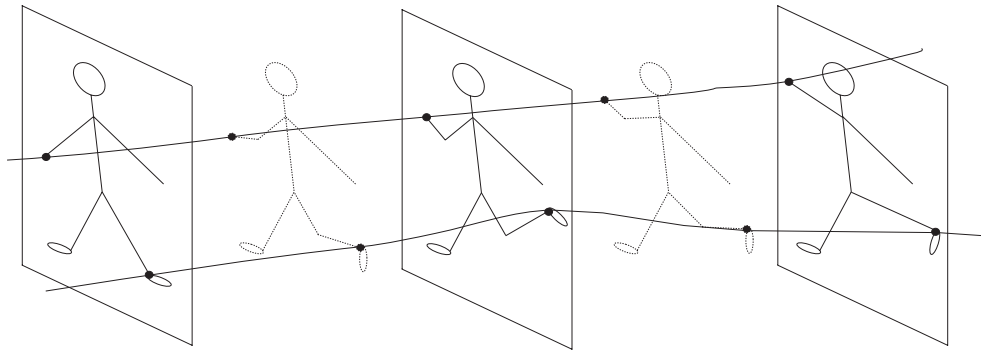
$$B = \begin{pmatrix} \frac{\phi_{\lambda_1}(x_1)}{\langle \phi_{\lambda_1}, \phi_{\lambda_1} \rangle_M} & \cdots & \frac{\phi_{\lambda_1}(x_{|F_M|})}{\langle \phi_{\lambda_1}, \phi_{\lambda_1} \rangle_M} \\ \vdots & \ddots & \vdots \\ \frac{\phi_{\lambda_n}(x_1)}{\langle \phi_{\lambda_n}, \phi_{\lambda_n} \rangle_M} & \cdots & \frac{\phi_{\lambda_n}(x_N)}{\langle \phi_{\lambda_n}, \phi_{\lambda_n} \rangle_M} \end{pmatrix}.$$

This would save valuable computation time, especially for large  $M$ , since the coefficients  $b_\lambda$  are easily calculated as a matrix product  $\vec{b} = B \cdot \vec{f}^t$ .

Moreover, the matrix  $B$  does not depend on the function that is to be expanded into series, therefore it need to be calculated only once for each  $M$  and can be repeatedly used.

## 6. Concluding remarks

- Each of the transforms described here is based on a compact semisimple Lie group of rank 3. All seven types of such Lie groups were considered. Our goal was to provide the tools for the expansion of functions of three variables given on a bounded region  $D$  of an Euclidean space  $\mathbb{R}^3$ . The variables can be either continuous or discrete (lattice grid points). The symmetry of the lattice is the Weyl group of the Lie group. The bounded region  $D$  has to be scaled to fit into the fundamental region  $F$  of the corresponding Lie group. In the case of functions given on a lattice grid, the scaling has to be accompanied



**Figure 8.** One of the possible applications of orbit functions is the construction of unknown transitional data (smooth interpolation). The figure shows how additional frames could be added to a film. In much the same way, the continuous deformation of the picture can be proceed and a three-dimensional image can be created from corresponding two-dimensional layers or cuts.

with the matching density of the grid points in  $F$ . Fortunately, the formalism admits choosing any density one may need. The scaling resulting in the inclusion  $D \subset F$  is not unique. Various options may be considered for specific functions.

- The uncommon special functions of our transforms are defined for compact semisimple Lie groups of any type and rank [22]. Their continuous and discrete orthogonality in  $F$  is assured [21]. Unlike the translation symmetry required in traditional Fourier expansions, the symmetry group of  $C$ -,  $S$ - and  $E$ -functions is the appropriate affine Weyl group, which contains the translations in  $\mathbb{R}^3$  as a subgroup.
- The uniformity of our approach, as to the type of the rank 3 Lie group, is illustrated here by considering the seven cases in parallel. The price paid for uniformity is the exploitation of non-orthogonal bases,  $\alpha$ ,  $\omega$ -bases and their duals whenever necessary. The majority of practically useful digital data usually given on cubic/square lattices with the simplest symmetry group. Only more costly experimental installations may use denser lattice arrangements of data collectors.

It should be pointed out that, at least for one type of transform, it is possible to avoid paying the price i.e. of having to work with orthonormal bases. The Weyl group of  $SU(n)$  is isomorphic to the permutation group  $S_n$  of  $n$  elements. Recently introduced transforms [14–17], based on  $S_n$  and on its alternating subgroup, exploit orthonormal bases in  $\mathbb{R}^n$ , although even there, the corresponding fundamental regions do not have orthogonal adjacent faces, in general.

- There is an additional freedom of choice whenever the underlying Lie group is not simple. Suppose that group is a product of two simple Lie groups,  $G^1 \times G^2$ . The fundamental region is then the Cartesian product of  $F(G^1)$  and  $F(G^2)$ .

For the expansion of class functions on the product group, we can combine  $C$ -functions on one with  $S$ -functions on the other. Similarly, we can combine  $C$ - or  $S$ -functions with  $E$ -functions, enlarging appropriately the fundamental region of the  $E$ -functions.

In much the same way, discretization can proceed differently on  $F(G^1)$  then on  $F(G^2)$ . The corresponding integers  $M_1$ ,  $M_2$  that fix it can be as different as one desires. Thus the density of grid points in  $F(G^1) \times F(G^2)$  may be very different on the two orthogonal components.

- A number of other properties of the orbit functions may prove to be useful (see [11–13] and references therein). Let us point out that each of the three types of functions split into mutually exclusive congruence classes. For a given semisimple Lie group, the number of congruence classes equals the order of the center of the Lie group.
- The possibility to introduce the  $C$ -,  $S$ - and  $E$ -functions by summation over a finite noncrystallographic Coxeter groups instead of the Weyl group of a Lie group appears to be rather interesting. In 3D there is just one such group  $H_3$ , the icosahedral group of order 120. Most of the properties carry over to this case in a simple straightforward way. The exception is the orthogonality, continuous or discrete. There is an analog of the fundamental region, but no lattice. Its role, perhaps, should be played by some quasicrystal?
- Orbit functions of this paper are related to the orthogonal polynomials [3], both being built using orbits of the Coxeter groups. It would be interesting to explore such a relation in detail.

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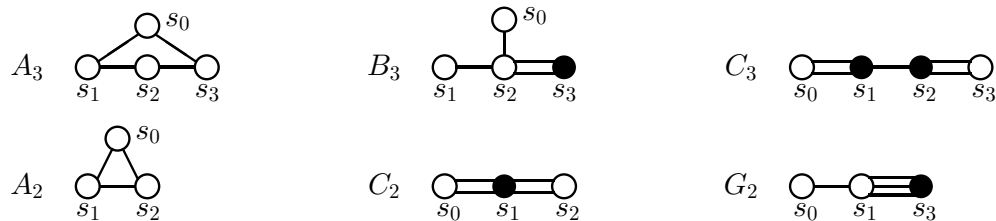
**Appendix A. Coefficients  $\varepsilon(x)$  for discrete scalar products**

Here we provide the coefficients  $\varepsilon(x)$ , required for the discrete scalar product and orthogonality of orbit functions. The coefficients are labeled by the points  $x$  of the discrete grid in  $F_M$ . Their value is equal to the number of points conjugate to  $x$  under the action of the corresponding Weyl group  $W$  on the maximal torus. Therefore the coefficients are determined by the stabilizer of  $x$  in the affine  $W$ . More precisely,

$$\varepsilon_x = \frac{|W|}{|\text{Stab}_W(x)|}.$$

Note that  $\varepsilon(x)$  does not depend on the density of the grid in  $F_M$ .

Order of the stabilizer  $|\text{Stab}_W(x)|$  on the maximal torus is calculated using extended Dynkin diagrams.



The calculation procedure is as follows. For a fixed grid point  $x$  of  $F_M$  ( $x = (\frac{s_1}{M}, \frac{s_2}{M}, \frac{s_3}{M})$  in  $\omega$ -basis) we consider the set  $[s_0, s_1, s_2, s_3]$  (such that  $s_0, s_1, s_2, s_3 \in \mathbb{Z}^{\geq 0}$  and  $s_0 + m_1s_1 + m_2s_2 + m_3s_3 = M$ , where  $m_i$ 's are the coefficients of the highest root in  $\alpha$ -basis)

**Table 1.** The values of  $\varepsilon(x)$  for the rank 3 simple Lie algebras. Assuming  $s_0, s_1, s_2, s_3 > 0$ .

$x$	$A_3$		$B_3$		$C_3$	
	$ \text{Stab}_W(x) $	$\varepsilon(x)$	$ \text{Stab}_W(x) $	$\varepsilon(x)$	$ \text{Stab}_W(x) $	$\varepsilon(x)$
$[s_0, s_1, s_2, s_3]$	1	24	1	48	1	48
$[s_0, s_1, s_2, 0]$	2	12	2	24	2	24
$[s_0, s_1, 0, s_3]$	2	12	2	24	2	24
$[s_0, 0, s_2, s_3]$	2	12	2	24	2	24
$[0, s_1, s_2, s_3]$	2	12	2	24	2	24
$[s_0, s_1, 0, 0]$	6	4	8	6	8	6
$[s_0, 0, s_2, 0]$	4	6	4	12	4	12
$[0, s_1, s_2, 0]$	6	4	4	12	4	12
$[s_0, 0, 0, s_3]$	6	4	6	8	6	8
$[0, s_1, 0, s_3]$	4	6	6	8	4	12
$[0, 0, s_2, s_3]$	6	4	4	12	8	6
$[s_0, 0, 0, 0]$	24	1	48	1	48	1
$[0, 0, s_2, 0]$	24	1	8	6	16	3
$[0, s_1, 0, 0]$	24	1	48	1	16	3
$[0, 0, 0, s_3]$	24	1	24	2	48	1

**Table 2.** The values of  $\varepsilon(x)$  for the rank 2 simple Lie algebras. Assuming  $s_0, s_1, s_2 > 0$ .

$x$	$A_2$		$C_2$		$G_2$	
	$ \text{Stab}_W(x) $	$\varepsilon(x)$	$ \text{Stab}_W(x) $	$\varepsilon(x)$	$ \text{Stab}_W(x) $	$\varepsilon(x)$
$[s_0, s_1, s_2]$	1	6	1	8	1	12
$[s_0, s_1, 0]$	2	3	2	4	2	6
$[s_0, 0, s_2]$	2	3	2	4	2	6
$[0, s_1, s_2]$	2	3	2	4	2	6
$[s_0, 0, 0]$	6	1	8	1	12	1
$[0, s_1, 0]$	6	1	4	2	4	2
$[0, 0, s_2]$	6	1	8	1	6	3

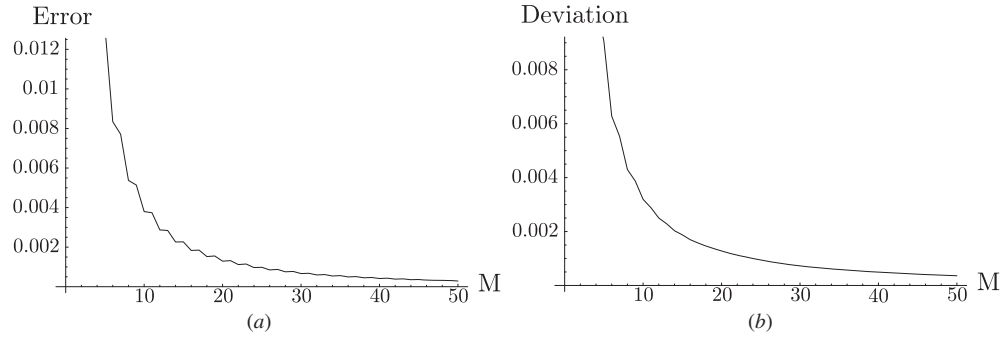
**Table 3.** The values of  $\varepsilon(x)$  for the rank 1 simple Lie algebra  $A_1$ . Assuming  $s_0, s_1 > 0$ .

$x$	$A_1$	
	$ \text{Stab}_W(x) $	$\varepsilon(x)$
$[s_0, s_1]$	1	2
$[s_0, 0]$	2	1
$[0, s_1]$	2	1

as attached to the corresponding nodes of the extended Dynkin diagram. Then we take the subdiagram with attached zeros only. It gives a Weyl group of order  $|\text{Stab}_W(x)|$ .

Specific values of the coefficients  $\varepsilon(x)$  are shown in tables 1–3 for the simple Lie groups of ranks 1, 2 and 3.

**Remark 5.** There are four semisimple Lie algebras of rank 3 which are not simple, namely  $A_1 \times A_1 \times A_1, A_2 \times A_1, C_2 \times A_1$  and  $G_2 \times A_1$ . The coefficients  $\varepsilon(x)$  in this cases are the products of the corresponding coefficients of simple components.



**Figure B1.** (a) The error (B.3) of the C-transform (B.2) as function of  $M$ ; (b) the standard deviation of the difference between the Gaussian function (B.1) and its interpolation (B.2) as function of  $M$ .

**Example 9.** Consider the case of  $A_1 \times A_1 \times A_1$ . The region  $F_M$  is a cube. The orthonormal axes correspond to the three  $A_1$ 's. Density of grid points along the three axes is determined by our choice of three positive integers, say  $M, M'$  and  $M''$ .

A point  $x$  of the grid in  $F_{M,M',M''}$  is given by

$$x = [s_0, s_1][s'_0, s'_1][s''_0, s''_1] = \left( \frac{s_1}{M}, \frac{s'_1}{M'}, \frac{s''_1}{M''} \right), \quad s_0, s_1, s'_0, s'_1, s''_0, s''_1 \in \mathbb{Z}^{\geq 0}$$

such that

$$M = s_0 + s_1, \quad M' = s'_0 + s'_1, \quad M'' = s''_0 + s''_1.$$

The coefficient  $\varepsilon(x)$  of  $A_1 \times A_1 \times A_1$  in this case is the product of the three corresponding coefficients of  $A_1$  which are read from the  $A_1$  table,  $\varepsilon(x) = \varepsilon_{x_s} \varepsilon_{x_{s'}} \varepsilon_{x_{s''}}$ .

### Appendix B. Example of C-transform on $A_1 \times A_1 \times A_1$

As an example, we chose to interpolate a discretization of a known function, namely the Gaussian function shown in equation (B.1)

$$g(x) = e^{-(x-p)^2} = e^{-(x-p) \cdot (x-p)}, \quad x \in \mathbb{R}^3, \tag{B.1}$$

where  $p \in F(A_1 \times A_1 \times A_1)$  is a fixed point inside the fundamental region.

The first test to be undertaken is to sample the function in the points of the grid  $F_M$  for several values of  $M$ . The continuous extension of  $g(x)$ , calculated from points of  $F_M$ , is

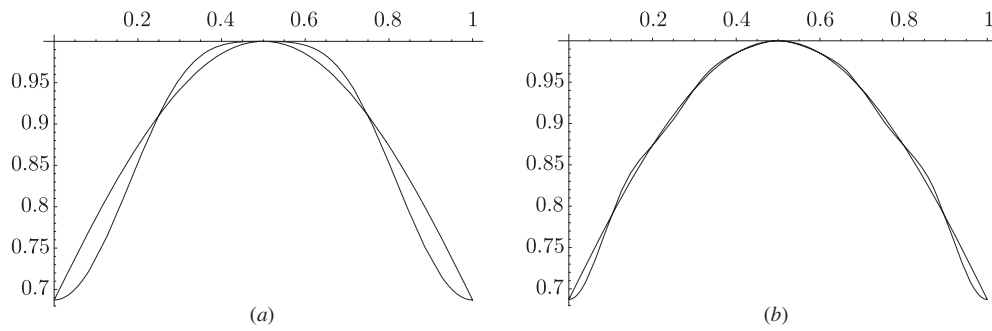
$$T(x) = \sum_{\lambda \in P} b_\lambda C_\lambda(x), \quad \text{with} \quad b_\lambda = \frac{\langle g, C_\lambda \rangle}{\langle C_\lambda, C_\lambda \rangle}. \tag{B.2}$$

For different values of  $M$ , figure B1(a) represents the error, as defined by the integral

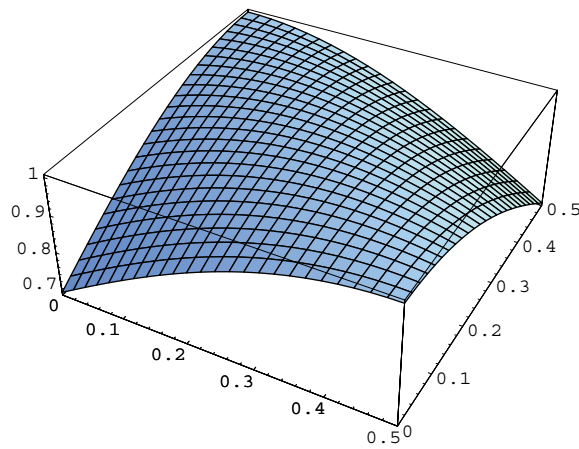
$$\int_F |T(x) - g(x)| dF. \tag{B.3}$$

The integral (24) was calculated using a simple Monte Carlo method with 10 000 randomly chosen points. We made sure that the granularity of the randomly generated points was much higher than that of the grid  $F_M$ .

Figure B1(b) shows the standard deviation of the sample set. It decreases in much the same way the error did in function of  $M$ .



**Figure B2.** Comparison of  $T(L(k))$  (B.2) with  $g(L(k))$  for  $k \in [0, 1]$  (B.4), with  $M = 4$  (a) and  $M = 10$  (b).



**Figure B3.** Two-dimensional cut (B.5) of the Gaussian function  $g(S(k, l))$ .

As a means to show what takes place visually, we give a plot of  $T$  as compared to  $g$  on the parametric line given by the transformation

$$L: \mathbb{R} \rightarrow \mathbb{R}^3, \quad k \xrightarrow{L} (k, k, k). \tag{B.4}$$

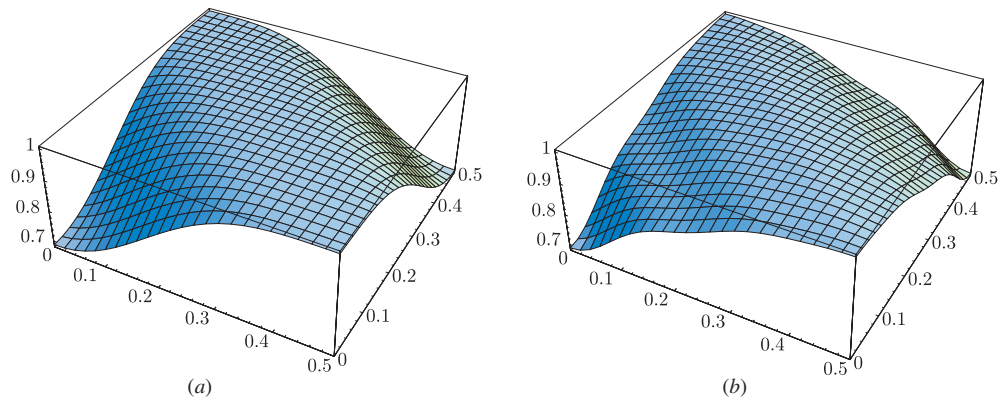
Figures B2(a) and (b) show the comparison between  $T(L(k))$  and  $g(L(k))$  for  $k \in [0, 1]$ , with  $M = 4$  and  $M = 10$ , respectively. As seen when comparing figures B2(a) and (b), the degree of oscillation increases as  $M$  increases, but the total error decreases. This is exactly as one would expect.

Consider two-dimensional cuts on the parametric surface given by

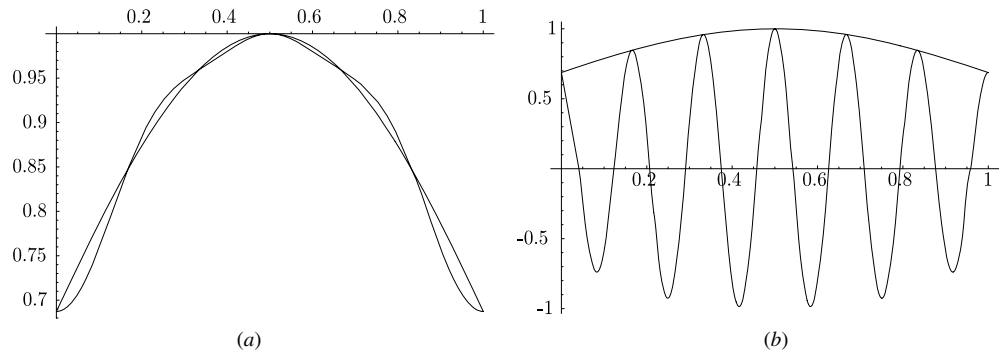
$$S: \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad (k, l) \xrightarrow{S} (k+l, k+k, k+l) \quad \text{with } k \in [0, \frac{1}{2}] \quad \text{and } l \in [0, \frac{1}{2}]. \tag{B.5}$$

A two-dimensional cut from the graphs of  $g$  is presented in figure B3. Figures B4(a) and (b) are two-dimensional cuts from the graphs of  $T$  with  $M = 4$  and  $M = 10$ , respectively.

In the case of  $A_1 \times A_1 \times A_1$  (as well as in the other cases, albeit less trivially), the set of pairwise orthogonal functions is not unique, i.e. one can scale all  $\lambda \in P$  by integer multiplier  $s \cdot (a, b, c)$  with  $(a, b, c) \in \mathbb{Z}^3, s \in \mathbb{Z}$ . This leads to a shifted system of orbit functions, that take the exact same values on the discrete grid  $F_M$ , but do in fact differ in their continuous



**Figure B4.** Two-dimensional cut (B.5) of the interpolation  $T(S(k, l))$  for  $M = 4$  (a) and  $M = 10$  (b).



**Figure B5.** Comparison between  $g(L(k))$  and  $T(L(k))$  (B.2) computed using the fundamental set of  $C$ -functions (a) or computed using the shifted by a factor  $(M, 0, 0)$  set of  $C$ -functions (b) for  $k \in [0, 1]$  (B.4), with  $M = 6$ .

behavior. Such a shifted system of functions can be understood as higher harmonics of the original functions on the grid  $F_M$ .

Examples with original and shifted systems of orbit functions follow. Figure B5(a) shows  $g$  and  $T$  with  $M = 6$ , for  $T$  computed using the fundamental set of  $C$ -functions and figure B5(b) shows the same graph, but for a  $T$  that was computed using the set of  $C$ -functions shifted by a factor of  $(M, 0, 0)$ .

The result is enlightening. The case using the shifted set of orbit functions exhibits a degree of error far above what could be tolerated, due to the fact that the higher harmonics of the  $C$ -functions, as expected, oscillate more. This derogatory result should not, however, dismiss the use of such higher harmonics. The Gaussian function extrapolated here is very smooth, and should thus be extrapolated with a sum of smoother functions. But in real life applications, the data could be quite chaotic, thus the use of higher harmonics could be useful.

It is also possible not to shift the entire set, but only a specific subset, and at that, not all should be shifted by the same factor. Careful consideration must be taken, because the functions do not obey the simplistic rule that a shifted function is equal to its counterpart. In fact, the set of adjacent functions is paired according to a reflective symmetry, but not according to translation symmetry.

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